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ON RANGE SOBOLEV SPACES DEFINED BY CESÀRO-HARDY OPERATORS

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On range Sobolev spaces defined by Cesàro-Hardy operators

Sobre espacios rango de tipo Sobolev definidos por operadores de Cesàro-Hardy

Memoria presentada por

Luis Sánchez Lajusticia

para optar al Grado de

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Dirigida por los doctores

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La carrera de matemáticas, además de abrirme los ojos a un mundo de conceptos inimaginables y proporcionarme la satisfacción de sentir cómo campos aparentemente distintos acaban encajando o respondiendo a una misma naturaleza, me dió la oportunidad de conocer a Julia y Violeta, dos muy buenas amigas, que también ejemplifican que personas distintas acaban encajando (como las matemáticas, en parte por responder a una misma naturaleza). Con ellas he pasado muy buenos ratos compartiendo matemáticas y lo que no son matemáticas, y espero seguir pasando muchos más.

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visto crecer y cambiar, siempre a mejor, y nos seguiremos viendo crecer y cambiar (crecimiento y cambio, no se puede negar que subyacen a todo). Y porque no somos completos si nos falta algo, no podía faltarme la música, aunque cuando hubo que elegir, las matemáticas se impusieron. Y en ese camino de la música tambin conocí personas importantes para mí, Ángel, Chus y Luis Pedro, que pusieron su aportación a mi forma de ser y comprender el lenguaje de los sonidos.

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Luis Sánchez Lajusticia
Zaragoza, junio de 2019.

Para Pepa,
Lucía y María

Contents

Introducción

En 1915, G. H. Hardy intentaba encontrar una demostración elemental de la desigualdad de Hilbert. La desigualdad discreta que obtuvo pudo extenderse a la siguiente desigualdad continua:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq C_p \int_0^\infty f^p(x)dx, \qquad f \geq 0,$$

que fue enunciada en 1920 [H1] y demostrada en 1925 [H2]. Gran parte del desarrollo inicial de la desigualdad de Hardy puede encontrarse en el libro (clásico) [HLP], y detalles sobre su historia en ambas formas, discreta y continua, en [KuMP], por ejemplo.

Las generalizaciones y aplicaciones de esta fórmula son destacables. Muchos de los aspectos de su desarrollo pueden encontrarse en [KMP], [KuMP], [KuP] y [OK].

La desigualdad

$$\left(\int_0^\infty \left|\frac{1}{t}\int_0^t f(s)ds\right|^p dt\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty f^p(t)dt\right)^{\frac{1}{p}}, \tag{1}$$

que se tiene para $1 < p < \infty$ (véase [HLP, p.245]), implica que el *operador de Cesàro*, que denotamos $\mathcal{C}$ y se define

$$\mathcal{C}f(t) = \frac{1}{t}\int_0^t f(s)ds, \quad t > 0, \tag{2}$$

es un operador acotado en $L_p(\mathbb{R}^+)$ con $\|\mathcal{C}\| \leq \frac{p}{p-1}$ para $1 < p < \infty$. De hecho, es también conocido que, si $\nu > 0$,

$$\left(\int_0^\infty \left|\frac{\nu}{t^\nu}\int_0^t (t-s)^{\nu-1}f(s)ds\right|^p dt\right)^{\frac{1}{p}} \leq \frac{\Gamma(\nu+1)\Gamma(1-\frac{1}{p})}{\Gamma(\nu+1-\frac{1}{p})}\|f\|_p, \qquad f \in L_p(\mathbb{R}^+), \tag{3}$$

para $1 < p < \infty$ y la constante $\frac{\Gamma(\nu+1)\Gamma(1-\frac{1}{p})}{\Gamma(\nu+1-\frac{1}{p})}$ es óptima para esa desigualdad (ver [HLP, Theorem 329]). Una desigualdad dual es la siguiente

$$\left(\int_0^\infty \left|\nu\int_s^\infty \frac{(t-s)^{\nu-1}}{t^\nu}f(t)dt\right|^p ds\right)^{\frac{1}{p}} \leq \frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\nu+\frac{1}{p}\right)}\|f\|_p. \tag{4}$$

La constante $\frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\nu+\frac{1}{p}\right)}$ también es óptima para esta desigualdad.

De manera natural, las desigualdades (3) y (4) sugieren definir operadores acotados de $L_p(\mathbb{R}^+)$ en $L_p(\mathbb{R}^+)$, que denotaremos, para $f \in L_p(\mathbb{R}^+)$, por

$$\mathcal{C}_\nu(f) := \frac{\nu}{t^\nu}\int_0^t (t-s)^{\nu-1} f(s)ds, \quad \text{si } 1 < p \le \infty,$$

y

$$\mathcal{C}_\nu^*(f) := \nu \int_t^\infty \frac{(s-t)^{\nu-1}}{s^\nu} f(s)ds, \quad \text{si } 1 \le p < \infty.$$

Para $\nu = 1$, los operadores $\mathcal{C}_1 = \mathcal{C}$ o $\mathcal{C}_1^* = \mathcal{C}^*$, o sus análogos discretos, han recibido diferentes nombres. Como ejemplo, son llamados operadores de Hardy en [KuMP], [DS], operadores de Cesàro en [BHS], [Bo], [Mo1], [Mo2], operadores de Copson en [Mo1], [Mo2], entre otros artículos. Hay también versiones de los anteriores operadores en el plano complejo, incluso en el caso generalizado; véase [AS], [LMPS]. El estudio de tales operadores se centra habitualmente en problemas sobre su acotación en diversos espacios, espectro, interpolación, dominio óptimo, estudio de las isometrías asociadas... (véase por ejemplo [AP], [DS], [BS1], [BS2]). Aquí llamaremos a $\mathcal{C}_\nu$, $\mathcal{C}_\nu^*$ operadores de Cesàro-Hardy. Estamos interesados en espacios rango de esos operadores integrales, dotados con la norma imagen de los espacios L_p, y centrándonos de forma más precisa en el caso Hilbert. La motivación para este enfoque es doble: por un lado surge de las conexiones que estos operadores tienen con la integro-diferenciación fraccionaria, y por otro lado de su relación con el movimiento browniano fraccionario o con el ruido blanco.

En el estudio de las ecuaciones abstractas de Cauchy "mal planteadas", es decir, cuando la solución de la ecuación no viene regida por un C_0-semigrupo, son relevantes familias como los C-semigrupos o los semigrupos integrados, y homomorfismos como semigrupos de distribuciones. En [AK] se consideran semigrupos de distribuciones temperadas que tienen como dominios álgebras de convolución $\mathcal{T}_1^{(n)}(t^n)$ -en una notación diferente a la que aparece en [AK]- definidas, para $n \in \mathbb{N}$, como la completación del espacio de funciones test $C_c^\infty(\mathbb{R}^+)$ en la norma

$$\|f\|_{1,(n)} := \int_0^\infty |f^{(n)}(t)|t^n \, dt < \infty, \quad f \in C_c^\infty(\mathbb{R}^+). \tag{5}$$

(Álgebras similares en toda la recta real $\mathbb{R}$ han sido introducidas en [BE]). El álgebra de Banach $\mathcal{T}_1^{(n)}(t^n)$ admite una extensión a orden de derivación fraccionario $\nu > 0$ considerando cierta derivada fraccionaria (denotada por $W^\nu f$) en lugar de la derivada habitual $f^{(n)}$; véanse [Mi1] y [GM]. Esta extensión, denotada por $\mathcal{T}_1^{(\nu)}(t^\nu)$, es también un álgebra de Banach de convolución con numerosas aplicaciones relacionadas con cálculos funcionales, semigrupos integrados y teoría de cuasimultiplicadores regulares, véase [GM]. Propiedades específicas o aplicaciones de $\mathcal{T}_1^{(\nu)}(t^\nu)$ como álgebra de Banach han aparecido en numerosos artículos, entre ellos [GMR1], [GMR2], [GMSt], [GS]. Si reemplazamos la norma L_1 de $t^n f^{(n)}$ por la norma L_p, con $1 < p < \infty$, de $t^n f^{(n)}$ en (5), podemos definir el $\mathcal{T}_1^{(\nu)}(t^\nu)$-módulo de convolución $\mathcal{T}_p^{(\nu)}(t^\nu)$. Lo procedente entonces es encontrar propiedades y aplicaciones de estos espacios de manera similar al caso del álgebra. Un primer análisis en esa dirección se hace en [GMMS] para $\nu = n \in \mathbb{N}$ y $p = 2$.

Por otra parte, estas ideas se aplican en problemas abstractos de Cauchy locales, a saber, problemas del tipo

$$\begin{cases} u'(t) = Au(t) + x, \; 0 \leq t < \tau \\ u(0) = 0 \end{cases} \tag{6}$$

donde A es un operador lineal cerrado en un espacio de Banach X y $\tau > 0$. Es conocido (véase [AEK, Theorem 2.1] o [V, Theorem 3.1]) que si para todo $x \in X$ el problema tiene solución única $u \in C^1([0,\tau), X) \cap C([0,\tau), D(A))$ (donde $D(A)$ se dota con la norma del grafo), entonces A es el generador de un semigrupo fuertemente continuo. Esto significa que las soluciones, inicialmente obtenidas en $[0,\tau)$, admiten extensiones a $[0,\infty)$ sin pérdida de regularidad y, más aún, son (uniformemente) exponencialmente acotadas.

Resulta que el espacio $\mathcal{T}_p^{(\nu)}(t^\nu)$ puede ser obtenido, de forma alternativa, como espacio rango o imagen del operador $\mathcal{C}_\nu^*$ con dominio en $L_p(\mathbb{R}^+)$, con lo que $\mathcal{C}_\nu^*$ puede ser entendido bajo el punto de vista que dan la integro-diferenciación fraccionaria. Además, integrales y derivadas fraccionarias tienen aplicación en la teoría del movimiento browniano fractal (fBm por sus siglas en inglés) y sistemas "autosimilares" (v. g., [FP], [Hu], [M], [SL]), con lo que los operadores de Cesàro-Hardy y los espacios de Hilbert que definen, es decir $\mathcal{T}_2^{(\nu)}(t^\nu)$, $\nu > 0$, se insertan de esta manera en esa teoría. La imagen de la transformada de Laplace sobre $\mathcal{T}_2^{(\nu)}(t^\nu)$ da lugar a un espacio de Hilbert de funciones holomorfas en el semiplano $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ que admite una descripción sencilla y podría ser un modelo adecuado para tratar con el fBm de tipo Riemann-Liouville.

La estructura de la memoria de tesis es como sigue.

En el Capítulo 1 presentamos los operadores de Cesàro-Hardy $\mathcal{C}_\nu$, $\mathcal{C}_\nu^*$ ($\nu > 0$) y los usamos para definir los espacios $\mathcal{T}_p^{(\nu)}(t^\nu)$. Nos centramos en la relación de estos operadores con la integro-diferenciación fraccionaria y otras interesantes propiedades que tienen que ver con la transformada de Laplace $\mathcal{L}$. Una herramienta útil en este contexto es la expresión de los operadores como una caso particular de subordinación a un cierto grupo de isometrías, $(T_p(t))_{t\in\mathbb{R}}$.

Tras haber definido los espacios, es natural preguntarse por la acotación, representación como operadores resolvente y propiedades espectrales de los operadores de Cesàro-Hardy generalizados $\mathcal{C}_\nu$ y $\mathcal{C}_\nu^*$ actuando en los subespacios de Sobolev $\mathcal{T}_p^{(\nu)}(t^\nu)$. Respondemos a algunas preguntas sobre esos temas en el Capítulo 2, también para los espacios $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ en toda la recta $\mathbb{R}$, definidos a partir de $\mathcal{T}_p^{(\nu)}(t^\nu)$.

Después, en el Capítulo 3, nos centramos en el caso $p = 1$ y estudiamos el comportamiento del álgebra $\mathcal{T}_1^{(\nu)}(t^\nu\omega)$, donde ω es una función peso, analizando semejanzas y diferencias con el caso $L_1(\omega)$: damos el espectro, la transformada de Gelfand y el espacio de caracteres de $\mathcal{T}_1^{(\nu)}(t^\nu\omega)$ en el caso semisimple y estudiamos un álgebra de Banach de tipo radical definida como álgebra cociente. Describimos esta última álgebra como un álgebra de funciones y analizamos sus ideales cerrados y derivaciones.

En el Capítulo 4 estudiamos el caso Hilbert, $p = 2$. Resulta que $\mathcal{T}_2^{(\nu)}(t^\nu)$ es un espacio de Hilbert de núcleo reproductivo (RKHS, abreviadamente). Determinamos su núcleo y revisamos algunos aspectos de la teoría general de RKHS para $\mathcal{T}_2^{(\nu)}(t^\nu)$, destacando una clara relación entre este espacio y los espacios que surgen asociados al movimiento browniano fractal en teoría de la probabilidad. Esta relación se discute parcialmente en la Sección 4.2, en conexión con el cálculo fraccionario de Riemann-Liouville.

Para $1 \leq p < \infty$, los espacios $H_p^{(\nu)}(\mathbb{C}^+)$ de funciones holomorfas en $\mathbb{C}^+$, versiones complejas de los espacios $\mathcal{T}_p^{(\nu)}(t^\nu)$, se definen en la Sección 4.3. Para ello es necesaria una forma compleja del cálculo fraccionario, y esto se consigue a través de la expresión del operador $\mathfrak{C}_\nu^*$ subordinado al grupo $T_p(t)$. De manera formal, reemplazando las derivadas fraccionarias reales por derivadas fraccionarias complejas, los espacios $\mathcal{T}_p^{(\nu)}(t^\nu)$ y $H_p^{(\nu)}(\mathbb{C}^+)$ pueden identificarse. Más aún, para $p = 2$ hay una correspondencia de tipo Paley-Wiener en el sentido en que $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu)) = H_2^{(\nu)}(\mathbb{C}^+)$ donde $\mathcal{L}$ es la transformada de Laplace. De hecho, $H_2^{(\nu)}(\mathbb{C}^+)$ es un RKHS no sólo para $\nu > 1/2$ sino para todo $\nu > 0$, y su núcleo reproductivo K_ν puede expresarse en forma integral. El resultado tipo Paley-Wiener y la fórmula para el núcleo se dan en el Teorema 4.3.2. En la Sección 4.4 se demuestra que la función $K_{\nu,z} := K_\nu(\cdot, z)$ satisface la equivalencia $\|K_{\nu,z}\|_{2,(\nu)} \sim |z|^{-1/2}$, $z \in \mathbb{C}^+$, salvo constantes de acotación. Esta equivalencia (o acotación) es en cierta forma sorprendente, porque las acotaciones habituales de las normas de los núcleos $\kappa(x, y)$ en los ejemplos clásicos de funciones holomorfas en dominios Ω suelen involucrar la distancia a la frontera del dominio Ω del punto $y \in \Omega$, con $\kappa_y := \kappa(\cdot, y)$, mientras que $\|K_{\nu,z}\|_{2,(\nu)}$ depende de la distancia *radial* de z, es decir, de z al origen, en $\mathbb{C}^+$.

Hemos considerado el operador $\mathcal{C}_\nu^*$ restringido a $L_2(\mathbb{R}^+)$ y su rango (o imagen) $\mathcal{T}_2^{(\nu)}(t^\nu)$, como el medio para mostrar las relaciones de los operadores de Cesàro-Hardy con el cálculo fraccionario y el movimiento browniano. Esta elección ha estado motivada por la fructífera relación de los espacios $\mathcal{T}_2^{(\nu)}(t^\nu)$ con las ecuaciones abstractas de Cauchy y sus familias asociadas de operadores. Como alternativa, podríamos haber elegido tomar el operador $\mathcal{C}_\nu$ y su rango $\mathcal{C}_\nu(L_2(\mathbb{R}^+))$ e intentar un tratamiento similar. El capítulo termina con la Sección 4.5, donde se muestra que $\mathcal{T}_2^{(\nu)}(t^\nu) = \mathcal{C}_\nu(L_2(\mathbb{R}^+))$, lo cual, en vista de las buenas y simples propiedades de los espacios $\mathcal{T}_2^{(\nu)}(t^\nu)$ y $H_2^{(\nu)}(\mathbb{C}^+)$ vistas en las secciones previas, sugiere la pregunta de si las operaciones de promedio fraccionario, como $\mathcal{C}_\nu$ hace, podrían ser de utilidad en la teoría browniana.

Para finalizar esta memoria, en el Capítulo 5 se abordan varias cuestiones sobre cómo generalizar los operadores y los espacios rango considerados previamente. Primero estudiamos la acotación de operadores de Cesàro-Hardy generalizados $\mathcal{C}_\kappa$, que escribimos utilizando producto de convolución $*$,

$$\mathcal{C}_\kappa(f) = \frac{1}{\chi_{(0,\infty)} * \kappa} f * \kappa$$

y nos preguntamos sobre qué condiciones deben cumplir esas funciones κ para dar lugar a operadores acotados (se recupera el operador generalizado clásico para la función $\kappa(t) = \mathfrak{r}_\nu(t) := t^{\nu-1}/\Gamma(\nu)$). Como consecuencia, se definen espacios rango correspondientes a esos operadores $\mathcal{C}_\kappa^*$, resultando ser módulos de Banach con respecto a las correspondientes álgebras de Banach, generalizando resultados previamente enunciados.

En la segunda parte del último capítulo nos centramos en los rangos de los operadores $\mathcal{C}_\kappa^*$ para establecer un marco de trabajo con aplicaciones a los problemas abstractos de Cauchy. Definimos homomorfismos de álgebras desde una nueva clase de funciones test y aplicamos nuestros resultados a operadores concretos. Se introduce la noción de semigrupos de κ-distribución para extender conceptos previos de semigrupos de distribuciones y para generalizar una fórmula de tipo Duhamel. Con estas herramientas, se obtiene un teorema sobre extensión de soluciones locales κ-convolucionadas (véase Teorema 5.2.17).

Introduction

In 1915, G. H. Hardy was trying to find out an elementary proof of the Hilbert inequality. The discrete inequality he obtained was extended to the continuous one:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq C_p \int_0^\infty f^p(x)dx, \qquad f \geq 0,$$

which was formulated in 1920 [H1] and proved in 1925 [H2]. Most of the early developments of Hardy inequality can be found in the classical book [HLP], and details about its history in both discrete and continuous forms in [KuMP], for instance.

The extensions and applications of this formula has been remarkable. Most important aspects of this development can be found in [KMP], [KuMP], [KuP] and [OK].

Inequality

$$(1) \qquad \left(\int_0^\infty \left|\frac{1}{t}\int_0^t f(s)ds\right|^p dt\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty f^p(t)dt\right)^{\frac{1}{p}},$$

that holds for $1 < p < \infty$ (see [HLP, p.245]), implies that the so-called Cesàro transformation $\mathcal{C}$, or *Cesàro operator*, defined by

$$(2) \qquad \mathcal{C}f(t) = \frac{1}{t}\int_0^t f(s)ds, \quad t > 0,$$

is a bounded operator on $L_p(\mathbb{R}^+)$ with $||\mathcal{C}|| \leq \frac{p}{p-1}$ for $1 < p < \infty$. In fact, it is also known that if $\nu > 0$ then

$$(3) \quad \left(\int_0^\infty \left|\frac{\nu}{t^\nu}\int_0^t (t-s)^{\nu-1} f(s)ds\right|^p dt\right)^{\frac{1}{p}} \leq \frac{\Gamma(\nu+1)\Gamma(1-\frac{1}{p})}{\Gamma(\nu+1-\frac{1}{p})}\|f\|_p, \qquad f \in L_p(\mathbb{R}^+),$$

for $1 < p < \infty$ and the constant $\frac{\Gamma(\nu+1)\Gamma(1-\frac{1}{p})}{\Gamma(\nu+1-\frac{1}{p})}$ is optimal in this inequality (it is also in [HLP, Theorem 329]). A closer (and dual) inequality is the following

$$(4) \qquad \left(\int_0^\infty \left|\nu\int_s^\infty \frac{(t-s)^{\nu-1}}{t^\nu} f(t)dt\right|^p ds\right)^{\frac{1}{p}} \leq \frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\nu+\frac{1}{p}\right)}\|f\|_p.$$

Also the constant $\frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\nu+\frac{1}{p}\right)}$ is optimal in the above inequality.

Naturally, inequalities (3) and (4) induce linear bounded operators from $L_p(\mathbb{R}^+)$ into $L_p(\mathbb{R}^+)$ that we denote for $f \in L_p(\mathbb{R}^+)$ by

$$\mathcal{C}_\nu(f) := \frac{\nu}{t^\nu}\int_0^t (t-s)^{\nu-1} f(s)ds, \text{ when } 1 < p \leq \infty,$$

and

$$\mathcal{C}^*_\nu(f) := \nu\int_t^\infty \frac{(s-t)^{\nu-1}}{s^\nu} f(s)ds, \text{ when } 1 \leq p < \infty.$$

For $\nu = 1$ operators $\mathcal{C}_1 = \mathcal{C}$ or $\mathcal{C}^*_1 = \mathcal{C}^*$, or their discrete counterparts, have received different names. As a sample, they are called Hardy's operators in [KuMP], [DS], Cesàro operators in [BHS], [Bo], [Mo1], [Mo2], Copson operators in [Mo1], [Mo2], among other papers. There are also versions of the above operators in the complex plane, even in the generalized case; see [AS], [LMPS]. The study of such operators is usually focused on problems around boundedness on diverse spaces, spectrum, interpolation, optimal domain, study of associated isometries, ...(see for instance [AP], [DS], [BS1], [BS2]). Here we call $\mathcal{C}_\nu$, $\mathcal{C}^*_\nu$ Cesàro-Hardy operators. We are interested in the range spaces, of these integral operators, endowed with the norm transferred from L_p spaces, and more precisely in the Hilbertian case L_2. The motivation for such an approach is two-fold, arising from the connections of those operators with fractional integro-differentation, from one side, and with fractional Brownian motion or white noise on the other hand.

In the study of "ill-posed" abstract Cauchy equations, so when the solution of the equation is not governed by a C_0-semigroup, families like C-semigroups or integrated semigroups, and homomorphisms like distribution semigroups are relevant. In [AK], tempered distribution semigroups are considered which have as domains convolution Banach algebras $\mathcal{T}_1^{(n)}(t^n)$ -in a different notation from that one of [AK]- defined, for $n \in \mathbb{N}$, as the completion of the space of test functions $C_c^\infty(\mathbb{R}^+)$ in the norm

$$\|f\|_{1,(n)} := \int_0^\infty |f^{(n)}(t)|t^n\, dt < \infty, \quad f \in C_c^\infty(\mathbb{R}^+). \tag{5}$$

(Similar algebras on the whole real line $\mathbb{R}$ had been introduced in [BE]). The Banach algebra $\mathcal{T}_1^{(n)}(t^n)$ admits an extension to fractional order of derivation $\nu > 0$ simply by considering certain fractional derivation (denoted by $W^\nu f$) instead of the usual derivation $f^{(n)}$; see [Mi1] and [GM]. This extension, denoted by $\mathcal{T}_1^{(\nu)}(t^\nu)$, is also a convolution Banach algebra which has a number of applications related to functional calculi, integrated semigroups and theory of regular quasimultipliers, see [GM]. Specific properties or applications of $\mathcal{T}_1^{(\nu)}(t^\nu)$ as a Banach algebra have been given in quite anumber of papers, among them [GMR1], [GMR2], [GMSt], [GS]. By replacing the L_1-norm of $t^n f^{(n)}$ with the L_p-norm, for $1 < p < \infty$, of $t^n f^{(n)}$ in (5), one defines the convolution Banach $\mathcal{T}_1^{(\nu)}(t^\nu)$-module $\mathcal{T}_p^{(\nu)}(t^\nu)$. It sounds sensible to find out properties and applications of such spaces similarly to the algebra case. A first analysis in that direction is done in [GMMS] for $\nu = n \in \mathbb{N}$ and $p = 2$.

On the other hand, this circle of ideas are of application in local abstract Cauchy problems. Namely, problems of the type

$$\begin{cases} u'(t) = Au(t) + x, \, 0 \leq t < \tau \\ u(0) = 0 \end{cases} \tag{6}$$

where A is closed linear operator on a Banach space X and $\tau > 0$. It is known (see [AEK, Theorem 2.1] or [V, Theorem 3.1]) that if for every $x \in X$, the problem has a unique solution $u \in C^1([0,\tau), X) \cap C([0,\tau), D(A))$ (where $D(A)$ is endowed with the graph norm), then A is the generator of a strongly continuous semigroup. This means that the solutions, initially obtained on $[0,\tau)$, admit extensions to $[0,\infty)$ without loss of regularity and moreover, are (uniformly) exponentially bounded.

It turns out that the space $\mathcal{T}_p^{(\nu)}(t^\nu)$ can be alternatively obtained as range space of the operator $\mathcal{C}_\nu^*$ with domain in $L_p(\mathbb{R}^+)$, so that $\mathcal{C}_\nu^*$ may well be regarded under the viewpoint that fractional integro-differentiation provides. On the other hand, fractional integrals and derivatives are of application in the theory of fractal Brownian motion (fBm, for short) and self-similar systems (v. g., [FP], [Hu], [M], [SL]), so that the Cesàro-Hardy operators and the Hilbertian spaces that they define, namely $\mathcal{T}_2^{(\nu)}(t^\nu)$, $\nu > 0$, appear in this way inserted in that theory. Then the action of the Laplace transform on $\mathcal{T}_2^{(\nu)}(t^\nu)$ gives rise to a Hilbert space of holomorphic functions on the half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ which admits a simple description and might be a suitable model in order to deal with fBm of Riemann-Liouville type.

The structure of this monograph is as follows.

In Chapter 1 we present the Cesàro-Hardy operators $\mathcal{C}_\nu$, $\mathcal{C}_\nu^*$ ($\nu > 0$) and use them to define the spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$. We focus on the relation of these operators with fractional integro-differentiation and other interesting properties involving the Laplace transform $\mathcal{L}$. A useful tool to do this is the expression of the operators as a particular case of subordination to a certain group of isometries, $(T_p(t))_{t\in\mathbb{R}}$.

After having defined the spaces, it is natural to ask for boundedness, representation as resolvent operators and spectral properties of the generalized Cesàro-Hardy operators $\mathcal{C}_\nu$ and $\mathcal{C}_\nu^*$ acting on the Sobolev subspaces $\mathcal{T}_p^{(\nu)}(t^\nu)$. We answer some questions about the above items in Chapter 2, also for Banach spaces $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ in the whole line $\mathbb{R}$ constructed out from $\mathcal{T}_p^{(\nu)}(t^\nu)$.

Then, in Chapter 3, we focus on the case $p = 1$ and study the behaviour of the algebra $\mathcal{T}_1^{(\nu)}(t^\nu\omega)$, where ω is a weight function, analyzing similarities and diferences with the case $L_1(\omega)$: we give the spectrum, Gelfand transform and character space of $\mathcal{T}_1^{(\nu)}(t^\nu\omega)$ in the semisimple case and study a radical Volterra type algebra introduced as quotient algebra. We describe such an algebra as a function algebra and discuss its closed ideals and derivations.

In Chapter 4 we study the Hilbert case $p = 2$. It turns out that $\mathcal{T}_2^{(\nu)}(t^\nu)$ is a reproducing kernel Hilbert space (RKHS, for short). The kernel is determined, and then some ingredients of the general theory of RKHS are revised for $\mathcal{T}_2^{(\nu)}(t^\nu)$, from which it becomes apparent that there exists a relationship between this space and spaces arising in fractal Brownian motion or probability theory. This relation is partly described in Section 4.2, in connection with the Riemann-Liouville fractional calculus.

For $1 \le p < \infty$, spaces $H_p^{(\nu)}(\mathbb{C}^+)$ of holomorphic functions in $\mathbb{C}^+$, complex versions of spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$, are introduced in Section 4.3. To do this, one needs a complex form of the fractional calculus, which is available through the subordination expression of the operator $\mathfrak{C}_\nu^*$ to the group $T_p(t)$. Formally, just replacing real fractional derivatives with complex fractional derivatives, spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$ and $H_p^{(\nu)}(\mathbb{C}^+)$ look identical. Moreover, for $p = 2$ there is a correspondence of Paley-Wiener type in the sense that $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu)) = H_2^{(\nu)}(\mathbb{C}^+)$ where $\mathcal{L}$ is the Laplace transform. Indeed, $H_2^{(\nu)}(\mathbb{C}^+)$ is a RKHS not only for $\nu > 1/2$ but for all $\nu > 0$, and its reproducing kernel K_ν can be expressed by means of a nice integral. The Paley-Wiener result and the formula of the kernel are given in Theorem 4.3.2. In Section 4.4, it is proved that the function $K_{\nu,z} := K_\nu(\cdot, z)$ satisfies the estimate $\|K_{\nu,z}\|_{2,(\nu)} \sim |z|^{-1/2}$, $z \in \mathbb{C}^+$, up to constants from below and from above. This estimate is somehow surprising since usual estimates for norms of kernels $\kappa(x, y)$ in classical examples of holomorphic functions on domains Ω involve the distance up to the boundary of the domain Ω of point $y \in \Omega$ of $\kappa_y := \kappa(\cdot, y)$, whereas $\|K_{\nu,z}\|_{2,(\nu)}$ depends on the *radial* distance of z, that is, of z to the origin, in $\mathbb{C}^+$.

We have taken the operator $\mathcal{C}_\nu^*$ restricted on $L_2(\mathbb{R}^+)$ and its range $\mathcal{T}_2^{(\nu)}(t^\nu)$, as the way to show the links of Cesàro-Hardy operators with fractional calculus and Brownian motion. This choice has been motivated by the fruitful relation of the spaces $\mathcal{T}_2^{(\nu)}(t^\nu)$ with abstract Cauchy equations and their associated operator families. Alternatively, we could have chosen to take the operator $\mathcal{C}_\nu$ and range $\mathcal{C}_\nu(L_2(\mathbb{R}^+))$ and try to make a similar treatment. The chapter finishes with Section 4.5, where we show that $\mathcal{T}_2^{(\nu)}(t^\nu) = \mathcal{C}_\nu(L_2(\mathbb{R}^+))$, which in view of the simple and good properties of spaces $\mathcal{T}_2^{(\nu)}(t^\nu)$ and $H_2^{(\nu)}(\mathbb{C}^+)$ seen in previous sections, suggests the question if averaging fractal operations, as $\mathcal{C}_\nu$ does, could be helpful in Brownian theory.

To finish the monograph, in Chapter 5 we approach some questions about generalizations of the operators and range spaces considered formerly. First we study the boundedness of generalized Cesàro-Hardy operators $\mathcal{C}_\kappa$, which we can write using convolution product $*$,

$$\mathcal{C}_\kappa(f) = \frac{1}{\chi_{(0,\infty)} * \kappa} f * \kappa$$

and then we ask about the conditions that must be given for these κ functions to generate bounded operators (we retrieve the classical generalized Cesàro-Hardy operators for the case $\kappa(t) = \mathfrak{r}_\nu(t) := t^{\nu-1}/\Gamma(\nu)$). As a consequence, range spaces corresponding to operators $\mathcal{C}_\kappa^*$ are defined, turning out to be Banach modules with respect to the correspondent Banach algebras, generalizing previously stated results.

In the second part of the last chapter we focus on the ranges of operators $\mathcal{C}_\kappa^*$ in order to establish a framework with application to abstract Cauchy problems. We define algebra homomorphisms from a new class of test-functions and apply our results to concrete operators. The notion of κ-distribution semigroups is introduced to extend previous concepts of distribution semigroups and to generalize a formula of Duhamel type. With these tools, a theorem about extensions of local κ-convoluted solutions is obtained (see Theorem 5.2.17).

Basic concepts and Notation

$\mathbb{N}$ is the set of natural numbers (starting at 1): $\mathbb{N} = \{1, 2, 3, \cdots\}$.
$\mathbb{R}$ is the set of real numbers: $\mathbb{R} = (-\infty, +\infty)$.
$\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}^- := (-\infty, 0)$.
$\mathbb{C}$ is the set of complex numbers.
$\mathbb{C}^+ = \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 0\}$
$\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$
Let R be $\mathbb{R}$ or $\mathbb{R}^+$. $C(R)$ is the set of continuous functions on R. $C_0(R)$ is the subset of $C(R)$ of functions vanishing at infinity, that is $\lim_{|x|\to\infty} f(x) = 0$.
Let I be an interval of real numbers. $C^{(\infty)}(I)$ is the set of infinitely differentiable functions on the interval I. In closed intervals $I = [a, b]$ (or semiclosed ones), it is understood that differentiability is just one-sided, e.g., from the right to a and from the left to b.
Let $n \in \mathbb{N}$. We will denote by $f^{(n)}$ the n-th order derivative of f.
Let S be a subset of $\mathbb{C}$. $C(S)$ is the set of continuous functions on S. $\mathcal{H}ol(S)$ is the set of holomorphic functions on S.
For a function f, $\operatorname{supp}(f)$ is the *support* of f.
$C_c^{(\infty)}[0, \infty)$ is the subset of $C^{(\infty)}([0, \infty))$ of functions with compact support in $[0, \infty)$.
$C_c^{(\infty)}(0, \infty)$ is the subset of $C^{(\infty)}((0, \infty))$ of functions with compact support in $(0, \infty)$.
Recall that $\mathcal{S}$, the Schwartz class over $\mathbb{R}$, is the subset of $C^{(\infty)}(\mathbb{R})$ of functions such that

$$\sup_{t\in\mathbb{R}} \left| t^m \frac{d^n}{dt^n} f(t) \right| < \infty$$

for all $m, n \in \mathbb{N} \cup \{0\}$.
We will denote by $\mathcal{S}_+$ the set of functions $\mathcal{S}_+ := \mathcal{S} \cap C^{(\infty)}[0, \infty)$.
Given $p \geq 1$, it is said that $q \geq 1$ is its *conjugate exponent* if $\frac{1}{p} + \frac{1}{q} = 1$. Note that $q = \frac{p}{p-1}$. For $p = 1$, we follow the usual convention $q = \infty$.
For $1 \leq p < \infty$ and $S \subseteq \mathbb{R}$, let $L_p(S)$ be the set of Lebesgue p-integrable (class of) functions, that is,

$$L_p(S) = \{f : S \to \mathbb{R} \text{ a.e. measurable } \ : \ \|f\|_p := \left(\int_S |f(t)|^p dt \right)^{\frac{1}{p}} < \infty\}.$$

Analogously,

$$L_p(\omega^p) = L_p(S, \omega^p) = \{f : S \to \mathbb{R} \text{ a.e. measurable } : \|f\|_p := \left(\int_S |f(t)|^p \omega^p(t) dt\right)^{\frac{1}{p}} < \infty\}.$$

for a given (nonnegative) function ω. Naturally, $\omega^p(t)$ will mean $(\omega(t))^p$. Note that, if we use for example $L_1(\omega)$, it will be clear from the context if we are working on $\mathbb{R}^+$ or on $\mathbb{R}$.
For the sake of completeness, recall that

$$L_\infty(S) = \{f : S \to \mathbb{R} \text{ a.e. measurable } : \|f\|_\infty < \infty\}.$$

where $\|f\|_\infty := \inf\{C \geq 0 \ : \ |f(t)| \leq C \text{ for almost every } t \in S\}$ is the *essential supremum* of f. $L_\infty(\omega)$ is the set of Lebesgue measurable (class of) functions such that

$$\|f\|_{\infty,\omega} := \operatorname{ess\,sup}_{t>0} |\omega(t) f(t)| < \infty.$$

$L^1_{loc}(S)$ is the set of a. e. measurable functions which are *locally integrable*, that is, its Lebesgue integral is finite on all compact subsets K of S.

$$L^1_{loc}(S) = \{f : S \to \mathbb{R} \text{ a.e. measurable } : f|_K \in L_1(S) \ \forall K \subset S, K \text{ compact}\}.$$

where $f|_K$ is the restriction of f to K.
For given $f, g \in L^1_{loc}(\mathbb{R}^+)$, we will denote by $f * g$ the usual (or Laplace) convolution product on $\mathbb{R}^+$, given by

$$(f * g)(t) = \int_0^t f(s) g(t-s) ds, \qquad t \geq 0.$$

We also follow the notation $\circ$ to denote the dual convolution product of $*$, given by

$$(f \circ g)(t) = \int_t^\infty f(s-t) g(s) ds, \qquad t \geq 0, \qquad f, g \in L_1(\mathbb{R}^+).$$

We define the cosine convolution product, $*_c$, as follows:

$$f *_c g := \frac{1}{2}\left(f * g + f \circ g + g \circ f\right), \qquad f, g \in L_1(\mathbb{R}^+).$$

We denote by $\mathcal{L}$ the usual Laplace transform for suitable functions f on $\mathbb{R}^+$,

$$\mathcal{L}f(z) = \int_0^\infty e^{-zt} f(t) dt, \quad z \in \mathbb{C}^+.$$

Γ is the Euler gamma function.
Recall that the Beta function, also called the Euler integral of the first kind, can be expressed by:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, \quad y > 0,$$

and satisfies the property $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
For a real number ν, $[\nu]$ is the integer part of ν.
Let $\sigma \in \mathbb{R}$. We will denote by Π_σ and $\overline{\Pi}_\sigma$ the sets

$$\Pi_\sigma := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > \sigma\}, \qquad \overline{\Pi}_\sigma := \{z \in \mathbb{C} \ : \ \operatorname{Re} z \geq \sigma\}.$$

Note that, with this notation, $\Pi_0 = \mathbb{C}^+$. If X is a Banach space, we will denote by $\mathcal{B}(X)$ the Banach algebra of bounded linear operators on X.
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y if the inclusion function between them is continuous, i.e. if there exists a constant $M \geq 0$ such that $\|x\|_Y \leq M\|x\|_X$ for every $x \in X$.
Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be three Banach spaces and $\odot$ a binary operation $\odot : X \times Y \to Z$. We write $X \odot Y \hookrightarrow Z$ to mean that there exists $M > 0$ such that $\|x \odot y\|_Z \leq M\|x\|_X \|y\|_Y$ for all $x \in X$ and $y \in Y$.
$\left(\int_a^b \pm \int_c^d\right) f(r)dr$ is just a shortened form to write $\int_a^b f(r)dr \pm \int_c^d f(r)dr$.
We denote by $\mathrm{L}_n^{(\alpha)}$ the generalized Laguerre polynomial of order n and $\alpha > -1$ given by

$$\mathrm{L}_n^{(\alpha)}(t) = \frac{t^{-\alpha}e^t}{n!}\frac{d^n}{dt^n}\left(t^{n+\alpha}e^{-t}\right) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{t^k}{k!}$$

and we denote by $\mathrm{L}_n = \mathrm{L}_n^{(0)}$ the usual Laguerre polynomials.
We denote by ${}_2F_1$ the Gaussian hypergeometric function,

$${}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.$$

where a, b, c are parameters which assume arbitrary real or complex values except for $c = 0, -1, -2, \dots$ and z is a complex variable.
For $z \in \mathbb{C}$, we will denote $e_z(t) = e^{-zt}$, $t \geq 0$.
For $\nu > 0$, we will denote

$$\mathfrak{r}_\nu(t) := \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad t > 0,$$

a function that will play an important role throughout the monograph. Note that $\mathfrak{r}_\nu(t)$ somehow encodes the same information as the so-called *Riesz kernels* or *Bochner-Riesz functions*:

$$R_s^{\nu-1}(t) := \begin{cases} \dfrac{(s-t)^{\nu-1}}{\Gamma(\nu)}, & \text{if } 0 \leq t < s; \\ 0, & \text{if } t \geq s. \end{cases}$$

That is, $R_s^{\nu-1}(t) = \mathfrak{r}_\nu(s-t)\chi_{(0,s)}(t)$, where χ_S is the *characteristic function* of the set S.

In many occasions throughout the monograph, we will use the variable constant convention, in which C denotes a constant which may not be the same in different lines. The constant is frequently written with subindexes, $C_{i_1,i_2,\cdots,i_n}$, to emphasize that it depends on some parameters or functions $i_1, i_2, \cdots, i_n$, other than the variable(s) involved. In general it can change in a chain of inequalities from one step to another, althought we do not change the notation. It can also assimilate constants, without changing its notation. We should not confuse these constants C with the Cesàro-Hardy operators, that we will denote by $\mathcal{C}$, $\mathcal{C}^*$, $\mathcal{C}_\nu$ and $\mathcal{C}_\nu^*$.
In this monograph, we consider that for a Banach algebra $\mathcal{A}$, the following inequality holds

$$\|f * g\|_{\mathcal{A}} \leq C\|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$$

where the constant C may not necessarily be 1.
Let X be a Banach space. X' is the topological dual space of X.
We denote by $\sigma(\Lambda)$ the usual spectrum of the operator Λ. Recall that the *spectrum of an operator* $\Lambda : D \to X$ is defined as

$$\sigma(\Lambda) := \{z \in \mathbb{C} \ : \ \text{ there is no bounded inverse operator } (zI - \Lambda)^{-1} : X \to D\}.$$

The *point spectrum of the operator* Λ is the set of eigenvalues of Λ (it is denoted here by $\sigma_\pi(\Lambda)$), it holds $\sigma_\pi(\Lambda) \subseteq \sigma(\Lambda)$, and the *resolvent set* of Λ is the complement of its spectrum, $\rho(\Lambda) = \mathbb{C} \setminus \sigma(\Lambda)$. Finally, the *resolvent operator for* $\mu \in$ is $R(\mu, \Lambda) := (\mu I - \Lambda)^{-1}$.

Chapter 1

Definitions and starting properties

In this chapter we present the (generalized) Cesàro-Hardy operators. First of all, we define a group of isometries that will allow us to express these operators as subordinated to that group. We take advantage of that subordination to derive some properties of the Cesàro-Hardy operators. Then we will use the operators to define $\mathcal{T}_p^{(\nu)}(t^\nu)$, a function space closely related to fractional derivation. We give also a set of properties about this space.

1.1 A group of isometries

1.1.1 Real case

Let $f\colon \mathbb{R}^+ \to \mathbb{C}$ be a complex, measurable function defined a. e. on the half-line $\mathbb{R}^+$. For $t \in \mathbb{R}$ and $1 \le p \le +\infty$, put

$$T_p(t)f(s) := e^{-t/p} f(e^{-t}s), \qquad \text{a.e.} \quad s > 0, \tag{1.1}$$

where t/∞ is understood with value 0.

Remarks 1.1.1. (1) Clearly, $t \mapsto T_p(t)$ is a group in $t \in \mathbb{R}$ acting by composition on functions f as above. Further, $(T_p(t))_{t\in\mathbb{R}}$ is a strongly continuous group of surjective isometries on $L_p(\mathbb{R}^+)$ if $1 \le p < \infty$, and on $C_0[0,\infty)$ if $p = \infty$. The isometric property of this group is fairly simple to check. As for the strong continuity, it is also part of folklore: for $1 \le p < \infty$, $h \in C_c(0,\infty)$ and $s, t \in \mathbb{R}$,

$$\|T_p(t)h - T_p(s)h\|_p^p = \int_0^\infty |e^{-t/p} h(e^{-t}r) - e^{-s/p} h(e^{-s}r)|^p \, dr$$

and so $\|T_p(t)h - T_p(s)h\|_p^p \to 0$ as $t \to s$, by the dominated convergence theorem, for example, since $\operatorname{supp}(h)$ is compact. For arbitrary f in $L_p(\mathbb{R}^+)$, one obtains $\|T_p(t)h - T_p(s)h\|_p^p \to 0$, as $t \to s$, using the density of $C_c(0,\infty)$ and the fact that $T_p(t)$, $t \in \mathbb{R}$, are isometries. The case $p = \infty$ is even simpler.

(2) As a matter of fact, the infinitesimal generator of $(T_p(t))_{t\in\mathbb{R}}$ is given by

$$(\Lambda_p f)(s) = -sf'(s) - \frac{1}{p}f(s)$$

(see Chapter 2, where this generator will be calculated in a more general situation).

1.1.2 Complex case

For $1 \le p < \infty$, let $H_p(\mathbb{C}^+)$ be the Hardy space on $\mathbb{C}^+$, which is formed by all holomorphic functions F in $\mathbb{C}^+$ such that

$$\|F\|_p := \sup_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(x+iy)|^p \, dy\right)^{1/p} < \infty.$$

Endowed with the norm $\|\cdot\|_p$, $H_p(\mathbb{C}^+)$ is a Banach space. Recall that the elements F of $H_p(\mathbb{C}^+)$ admit extension to the real line $i\mathbb{R}$ almost everywhere (and via nontangential limits) with $\|F\mid_{i\mathbb{R}}\|_p = \|F\|_p$. (It is because of this norm equality that we denote the norm in $H_p(\mathbb{C}^+)$ as $\|\cdot\|_p$, that is, the same notation we use for the norm in $L_p(i\mathbb{R})$ or $L_p(\mathbb{R}^+)$. The space will be clear in each situation).

The space $H_p(\mathbb{C}^+)$ can alternatively be described as follows. For $\theta \in (-\pi/2, \pi/2)$ and every function F holomorphic in $\mathbb{C}^+$, set $F_\theta(z) := F(ze^{i\theta})$, for $z \in \mathbb{C}^+$. Then $H_p(\mathbb{C}^+)$ is the space of holomorphic functions F on $\mathbb{C}^+$ such that

$$\|F\|_{p,rad} := \sup_{-\pi/2<\theta<\pi/2}\left(\frac{1}{2\pi}\int_0^{\infty} |F_\theta(r)|^p dr\right)^{1/p} < \infty.$$

Moreover, $\|\cdot\|_{p,rad}$ is a norm and

$$\|F\|_{p,rad} = \|F\|_p, \quad \forall\ F \in H_p(\mathbb{C}^+); \tag{1.2}$$

see [S] (which extends to arbitrary p the case $p=2$ proven in [Dz]).

The operator $T_p(t)$ extends obviously to functions F in the complex plane by putting

$$T_p(t)F(z) := e^{-t/p}F(e^{-t}z), \qquad \text{for all} \quad z \in \mathbb{C}, \quad t \in \mathbb{R}. \tag{1.3}$$

Remarks 1.1.2. (1) In particular, for $F \in H_p(\mathbb{C}^+)$, the function $T_p(t)F$ is holomorphic in $z \in \mathbb{C}^+$ and, for $1 \le p < \infty$,

$$\begin{aligned}\|T_p(t)F\|_p^p &= \sup_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t}|F(e^{-t}(x+iy))|^p \, dy\right) \\ &= \sup_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(e^{-t}x+iu)|^p \, du\right) = \|F\|_p^p,\end{aligned}$$

whence it follows that $T_p(t)$ is an isometry from $H_p(\mathbb{C}^+)$ onto itself for every $t \in \mathbb{R}$. Further,

$$\begin{aligned}\|T_p(t)F - T_p(s)F\|_p &= \|(T_p(t)F)\mid_{i\mathbb{R}} - (T_p(s)F)\mid_{i\mathbb{R}}\|_p \\ &= \|T_p(t)(F\mid_{i\mathbb{R}}) - T_p(s)(F\mid_{i\mathbb{R}})\|_p \to 0, \text{ as } t \to s,\end{aligned}$$

as above in the real case. In conclusion, $(T_p(t))_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $H_p(\mathbb{C}^+)$, if $1 \le p < \infty$.

(2) When $p = \infty$ we need to consider the Banach space $\mathcal{A}_0(\mathbb{C}^+)$ of all holomorphic functions F in $\mathbb{C}^+$ with continuous extension to $\overline{\mathbb{C}^+} = \mathbb{C}^+ \cup i\mathbb{R}$ and such that $\lim_{z\in\overline{\mathbb{C}^+},\ z\to\infty} F(z) = 0$. then it is readily seen in a similar way to above that $(T_\infty(t))_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $\mathcal{A}_0(\mathbb{C}^+)$.

(3) Let $1 \le p < \infty$ and $t \in \mathbb{R}$. Then the adjoint operator of $T_p(t)$ is given by

$$T_p(t)^* = T_q(-t) \tag{1.4}$$

with q conjugate exponent of p. In particular $(T_2(t))_{t\in\mathbb{R}}$ is a group of unitary operators on $L_2(\mathbb{R}^+)$ and $H_2(\mathbb{C}^+)$

1.1.3 The group $T_p(t)$ and the Laplace transform

We can apply the Laplace transform $\mathcal{L}$ to functions f on $\mathbb{R}^+$ such that fe_z is integrable (recall that $e_z(t) := e^{-zt}$). We have, just formally for the moment,

$$\begin{aligned}\mathcal{L}(T_p(t)f)(z) &= e^{-t/p}\int_0^\infty f(e^{-t}s)e^{-zs}\,ds \;=\; e^{t/q}\int_0^\infty f(r)e^{-ze^t r}\,dr\\ &= e^{t/q}(\mathcal{L}f)(e^t z) \;=\; T_q(-t)(\mathcal{L}f)(z),\end{aligned}$$

for $t \in \mathbb{R}$, $z \in \mathbb{C}^+$, $1 \le p \le \infty$ and q its conjugate exponent. In short,

$$\mathcal{L} \circ T_p(t) = T_q(-t) \circ \mathcal{L}, \qquad t \in \mathbb{R}.$$

In order to make the above relatonship rigorous, note that for every function $h \in C_c^{(2)}(\mathbb{R}^+)$ one has $h(t) = e^{-t}g(t)$ with $g \in C_c^{(2)}(\mathbb{R}^+)$ so that for every $z \in \mathbb{C}^+$,

$$|\mathcal{L}h(z)| = |(\mathcal{L}g)(z+1)| = \frac{|(\mathcal{L}g'')(z+1)|}{|(z+1)|^2} \le \frac{|(\mathcal{L}(|g''|)(1)}{|(z+1)|^2}.$$

Hence

$$\begin{aligned}\|\mathcal{L}h\|_r &\le (\mathcal{L}|g''|)(1)\sup_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{((x+1)^2+y^2)^r}\right)^{1/r}\\ &= (\mathcal{L}|g''|)(1)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{((x+1)^2+y^2)^r}\right)^{1/r} < \infty;\end{aligned}$$

that is, $\mathcal{L}h \in H_r(\mathbb{C}^+)$ for all $1 \le r \le \infty$. Thus one has

$$\mathcal{L}(T_p(t)f) = T_q(-t)(\mathcal{L}f), \quad t \in \mathbb{R},\ \forall f \in C_c^{(2)}(\mathbb{R}^+).$$

Certainly, the above equality holds for more class of functions. For example, let p be such that $1 \le p \le 2$. Then Hausdorff-Young's inequality tells us that the Fourier transform $\mathcal{F}$ is bounded (it is a contraction, indeed) from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$. Now, for $x > 0$ and $f \in L_p(\mathbb{R}^+)$ we have $\mathcal{L}f(x + i\,\cdot\,) = \mathcal{F}(e^{-x\cdot}f) \in L_q(\mathbb{R})$ with $\|\mathcal{F}(e^{-x\cdot}f)\|_q \le \|e^{-x\cdot}f\|_p \le \|f\|_p$ (with the corresponding version when $q = \infty$). This means that $\mathcal{L}f \in H_q(\mathbb{C}^+)$. So we have

Lemma 1.1.3. *For $1 \leq p \leq 2$, q such that $\frac{1}{p} + \frac{1}{q} = 1$ and $t \in \mathbb{R}$, the following diagram is commutative:*

$$\begin{array}{ccc} L_p(\mathbb{R}^+) & \xrightarrow{T_p(t)} & L_p(\mathbb{R}^+) \\ \mathcal{L}\big\downarrow & & \big\downarrow\mathcal{L} \\ H_q(\mathbb{C}^+) & \xrightarrow{T_q(-t)} & H_q(\mathbb{C}^+) \end{array}$$

That is,

(1.5) $$\mathcal{L}(T_p(t)f) = T_q(-t)(\mathcal{L}f), \quad f \in L_p(\mathbb{R}^+).$$

1.2 Generalized Cesàro-Hardy operators

1.2.1 Real case: Cesàro-Hardy operators on the real line

Let $\nu > 0$. For a measurable function f defined a. e. on $\mathbb{R}^+$ set

(1.6) $$\mathcal{C}_\nu f(s) := \frac{\nu}{s^\nu} \int_0^s (s-u)^{\nu-1} f(u)\, du, \quad s > 0.$$

(1.7) $$\mathcal{C}_\nu^* f(s) := \nu \int_s^\infty \frac{(u-s)^{\nu-1}}{s^\nu} f(u)\, du, \quad s > 0.$$

Operators $\mathcal{C}_\nu$, $\mathcal{C}_\nu^*$ induced by formulas (1.6) and (1.7) are called (generalized) Cesàro-Hardy operators in this work. Usually, $\mathcal{C}_\nu$ is taken as starting point and $\mathcal{C}_\nu^*$ is then presented as the adjoint operator of $\mathcal{C}_\nu$. As we said in the Introduction, for $\nu = 1$, $\mathcal{C}_1$ is called sometimes the Hardy operator, and $\mathcal{C}_1^*$ the Copson operator. More precisely, take f in $\mathcal{S}_+$. Hardy's inequalities say that

$$\|\mathcal{C}_\nu f\|_r \leq A_{\nu,r} \|f\|_r, \quad 1 < r \leq \infty,\ f \in L_r(\mathbb{R}^+),$$

and

$$\|\mathcal{C}_\nu^* f\|_r \leq B_{\nu,r} \|f\|_r, \quad 1 \leq r < \infty,\ f \in L_r(\mathbb{R}^+),$$

with constants $A_{\nu,r} = \Gamma(\nu+1)\Gamma(1-(1/r))\Gamma(\nu+1-(1/r))^{-1}$ and $B_{\nu,r} = \Gamma(\nu+1)\Gamma(1/r)\Gamma(\nu+(1/r))^{-1}$; see [HLP, p. 245]. Thus the integrals given in (1.6) and (1.7) define linear bounded operators $\mathcal{C}_\nu \colon L_r(\mathbb{R}^+) \to L_r(\mathbb{R}^+)$, $1 < r \leq \infty$, and $\mathcal{C}_\nu^* \colon L_r(\mathbb{R}^+) \to L_r(\mathbb{R}^+)$, $1 \leq r < \infty$, respectively.

In this monograph, we initially focus on the operator $\mathcal{C}_\nu^*$ rather than on $\mathcal{C}_\nu$. Let p be $1 < p < \infty$ and q its conjugate exponent. It is readily seen, using Fubini's theorem, that $\mathcal{C}_\nu^* \colon L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+)$ is the adjoint operator of $\mathcal{C}_\nu \colon L_q(\mathbb{R}^+) \to L_q(\mathbb{R}^+)$. Moreover, the bounded operator $\mathcal{C}_\nu^* \colon L_1(\mathbb{R}^+) \to L_1(\mathbb{R}^+)$ can be regarded as the restriction operator to $L_1(\mathbb{R}^+)$ of the adjoint $(\mathcal{C}_\nu\, |_{C_0(\mathbb{R}^+)})^* \colon \mathcal{M}(\mathbb{R}^+) \to \mathcal{M}(\mathbb{R}^+)$. Here $\mathcal{M}(\mathbb{R}^+)$ is the Banach space of bounded regular Borel measures on $\mathbb{R}^+$ and $\mathcal{C}_\nu\, |_{C_0(\mathbb{R}^+)}$ is the restriction of the bounded operator $\mathcal{C}_\nu \colon L_\infty(\mathbb{R}^+) \to L_\infty(\mathbb{R}^+)$ to $C_0(\mathbb{R}^+)$.

1.2.2 Cesàro-Hardy operators as subordinated to semigroups

For $\nu > 0$, $1 \le p < \infty$, $f \in L_p(\mathbb{R}^+)$ and $s > 0$ we have

$$\begin{aligned} \mathcal{C}_\nu^* f(s) &= \nu \int_s^\infty (r-s)^{\nu-1} r^{-\nu} f(r)dr \;=\; \nu \int_0^\infty (1-e^{-t})^{\nu-1} f(e^t s)dt \\ &= \nu \int_0^\infty (1-e^{-t})^{\nu-1} e^{-t/p} (T_p(-t)f)(s)dt \end{aligned}$$

where we have used the change of variable $r = se^t$ in the second equality. Also, by (1.4) for $1 < q \le \infty$, $g \in L_q(\mathbb{R}^+)$ and $s > 0$,

$$\mathcal{C}_\nu g(s) \;=\; \nu \int_0^\infty (1-e^{-t})^{\nu-1} e^{-t/p} (T_q(t)g)(s)dt$$

where p and q are conjugate exponents.

In view of the above, put $\varphi_{\nu,p}(t) := \nu(1-e^{-t})^{\nu-1} e^{-t/p}$, $t > 0$, for every $\nu > 0$ and $1 \le p < \infty$. Then $\varphi_{\nu,p} \in L_1(\mathbb{R}^+)$ and therefore we have proved the following result.

Proposition 1.2.1. *For $\nu > 0$, $1 \le p < \infty$, q the conjugate exponent of p, $f \in L_p(\mathbb{R}^+)$ and $g \in L_q(\mathbb{R}^+)$ we have*

$$\mathcal{C}_\nu^* f = \int_0^\infty \varphi_{\nu,p}(t) T_p(-t) f dt, \quad \mathcal{C}_\nu g = \int_0^\infty \varphi_{\nu,p}(t) T_q(t) g dt, \tag{1.8}$$

where the convergence of integrals are in the Bochner sense.

The proposition tells us that $\mathcal{C}_\nu^*$ and $\mathcal{C}_\nu$ are given by subordination with respect to the semigroups $(T_p(-t))_{t\ge 0}$, $(T_q(t))_{t\ge 0}$, respectively, in terms of the Hille-Phillips operational calculus. This property had been observed for $\nu = 1$ in [AS] and for $\nu > 0$ in [LMPS] under slighty different expressions.

Corollary 1.2.2. *For $\nu, \mu > 0$, $1 < p < \infty$ and $t \in \mathbb{R}$,*

(1) $\quad T_p(t)\mathcal{C}_\nu^* = \mathcal{C}_\nu^* T_p(t)$ *and* $T_p(t)\mathcal{C}_\nu = \mathcal{C}_\nu T_p(t)$ *on* $L_p(\mathbb{R}^+)$.

(2) $\quad \mathcal{C}_\nu^* \circ \mathcal{C}_\mu = \mathcal{C}_\mu \circ \mathcal{C}_\nu^*$ *on* $L_p(\mathbb{R}^+)$.

Proof. (1) Let $\nu > 0$, $f \in L_p(\mathbb{R}^+)$ with $1 < p < \infty$ and $t \in \mathbb{R}$.

$$\begin{aligned} T_p(t)(\mathcal{C}_\nu^* f) &= T_p(t)\left(\int_0^\infty \varphi_{\nu,p}(s) T_p(-s) f ds\right) = \int_0^\infty \varphi_{\nu,p}(s) T_p(t) T_p(-s) f ds \\ &= \int_0^\infty \varphi_{\nu,p}(s) T_p(-s) T_p(t) f ds \;=\; \mathcal{C}_\nu^*(T_p(t)f), \end{aligned}$$

where we have used that $(T_p(t))_{t\in\mathbb{R}}$ is a group. The second equality is analogous.

(2) Let $\nu, \mu > 0$, $1 < p < \infty$ and $f \in L_p(\mathbb{R}^+)$. Since $(T_p(t))_{t\in\mathbb{R}}$ is a group,

$$\begin{aligned}
(\mathcal{C}_\nu^* \circ \mathcal{C}_\mu)f &= \int_0^\infty \varphi_{\nu,p}(t)T_p(-t)\left(\int_0^\infty \varphi_{\mu,q}(s)T_p(s)f ds\right) dt \\
&= \int_0^\infty \int_0^\infty \varphi_{\nu,p}(t)\varphi_{\mu,p}(s)T_p(-t)T_p(s)f dsdt \\
&= \int_0^\infty \int_0^\infty \varphi_{\mu,p}(s)\varphi_{\nu,p}(t)T_p(s)T_p(-t)f dsdt \\
&= \int_0^\infty \varphi_{\mu,p}(s)T_p(s)ds\left(\int_0^\infty \varphi_{\nu,p}(t)T_p(-t)f dt\right) ds \;=\; (\mathcal{C}_\mu \circ \mathcal{C}_\nu^*)f,
\end{aligned}$$

as we wanted to show. □

1.2.3 Complex case: Cesàro-Hardy operators on the half plane

We introduce here generalized Cesàro-Hardy operators acting on $\mathbb{C}^+$ via the semigroup $T_p(-t)$ of isometries considered in the preceding subsection. In this way, we avoid tedious calculations to check the holomorphy of the integral functions involved.

Let $1 \le p < \infty$. Recall that $H_p(\mathbb{C}^+)$ is the Banach space of holomorphic functions F on $\mathbb{C}^+$ such that this norm (the norm of $H_p(\mathbb{C}^+)$) is finite:

$$\|F\|_p := \sup_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^\infty |F(x+iy)|^p \, dy\right)^{1/p} < \infty.$$

Definition 1.2.3. *For $\nu > 0$, $1 \le p < \infty$ and q its conjugate exponent, $F \in H_p(\mathbb{C}^+)$ and $G \in H_q(\mathbb{C}^+)$, define*

$$\mathfrak{C}_\nu^* F := \int_0^\infty \varphi_{\nu,p}(t)T_p(-t)F dt \quad \textit{and} \quad \mathfrak{C}_\nu G := \int_0^\infty \varphi_{\nu,p}(t)T_q(t)G dt$$

where the convergence of integrals are in the Bochner sense.

From the definition it is evident that $\mathfrak{C}_\nu^* F$ is holomorphic in $\mathbb{C}^+$, in fact $\mathfrak{C}_\nu^* F \in H_p(\mathbb{C}^+)$ and so in particular there exists the nontangencial limit $\mathfrak{C}_\nu^* F(iy) := \lim_{z\to iy} \mathfrak{C}_\nu^* F(z)$ for almost everywhere $y \in \mathbb{R}$.

Let us develop the vector valued integral in Definition 1.2.3. Take $z \in \mathbb{C}^+$, $z = |z|e^{i\theta}$ with $-\pi/2 \le \theta \le \pi/2$ and $F \in H_p(\mathbb{C}^+)$. Put $F_\theta(u) := F(ue^{i\theta})$, for $u > 0$. By (1.2) it follows that $F_\theta \in L_p(\mathbb{R}^+)$ for all $\theta \in [-\pi/2, \pi/2]$. Then

$$\begin{aligned}
\mathfrak{C}_\nu^* F(z) &= \int_0^\infty \varphi_{\nu,p}(t)T_p(-t)F(z)dt \;=\; \nu\int_0^\infty (1-e^{-t})^{\nu-1}T_p(-t)F(ze^t)dt \\
&= \nu\int_1^\infty u^{-\nu}(u-1)^{\nu-1}F_\theta(|z|u)du \\
&= \nu\int_{|z|}^\infty (r-|z|)^{\nu-1}r^{-\nu}F_\theta(r)dr \;=\; \mathcal{C}_\nu^* F_\theta(|z|),
\end{aligned}$$

where the latter operator $\mathcal{C}_\nu^*$ is the real one defined on $L_p(\mathbb{R}^+)$ as in (1.7). Note that when $\theta = -\pi/2$ or $\theta = \pi/2$ then $\mathfrak{C}_\nu^* F$ is defined only a. e. on $(-\infty, 0)$ or $(0, \infty)$, respectively.

One can also express $\mathfrak{C}_\nu^* F$ on $\mathbb{C}^+$ by a complex integral: For $z = |z|e^{i\theta}$ as above,

$$\begin{aligned}\mathfrak{C}_\nu^* F(z) &= \nu \int_{|z|}^{\infty} (re^{i\theta} - |z|e^{i\theta})^{\nu-1} (re^{i\theta})^{-\nu} F(re^{i\theta}) d(re^{i\theta}) \\ &= \nu \int_{|z|\cdot e^{i\theta}}^{\infty\cdot e^{i\theta}} \frac{(\lambda - z)^{\nu-1}}{\lambda^\nu} F(\lambda) d\lambda \;=\; \nu \int_{0\cdot e^{i\theta}}^{\infty\cdot e^{i\theta}} \lambda^{\nu-1} \frac{F(\lambda+z)}{(\lambda+z)^\nu} d\lambda.\end{aligned}$$

Here $\lambda \mapsto \lambda^\beta$ is defined taking the principal argument continuous in $\mathbb{C} \setminus (-\infty, 0]$.

Remark 1.2.4. It can be shown by standard methods that the above complex integral giving $\mathfrak{C}_\nu^* F$ on $\mathbb{C}^+$ is independent of the ray of integration; that is,

$$\mathfrak{C}_\nu^* F(z) = \nu \int_{0\cdot e^{i\omega}}^{\infty\cdot e^{i\omega}} \lambda^{\nu-1} \frac{F(\lambda+z)}{(\lambda+z)^\nu} d\lambda.$$

for every $z \in \mathbb{C}^+$ and every $\omega \in (-\pi/2, \pi/2)$. We will not use this property here.

It can be also shown, with similar arguments as those used before, that one has

$$\mathfrak{C}_\nu G(z) = \frac{\nu}{z} \int_{0\cdot e^{i\omega}}^{|z|e^{i\omega}} (z - \lambda)^{\nu-1} G(\lambda) d\lambda,$$

for G, z and ω as above, after Definition 1.2.3. We do notice this fact about the operator $\mathfrak{C}_\nu$ for the shake of completeness, but it will not be used in this monograph.

An immediate consequence of Definition 1.2.3 is that the complex Cesàro-Hardy operators commute.

Corollary 1.2.5. *For $\nu, \mu > 0$, $1 < p < \infty$ and $F \in H_p(\mathbb{C}^+)$,*

$$(\mathfrak{C}_\nu^* \circ \mathfrak{C}_\mu) F = (\mathfrak{C}_\mu \circ \mathfrak{C}_\nu^*) F.$$

Another interesting consequence of the subordination to the groups $T_r(t)$ is that the Laplace transform $\mathcal{L}$ intertwines Cesàro-Hardy operators.

Corollary 1.2.6. *Let $\nu > 0$. Then,*

(1) $\mathcal{L} \circ \mathcal{C}_\nu^* = \mathfrak{C}_\nu \circ \mathcal{L}$ *on $L_p(\mathbb{R}^+)$ for $1 \le p \le 2$.*

(2) $\mathcal{L} \circ \mathcal{C}_\nu = \mathfrak{C}_\nu^* \circ \mathcal{L}$ *on $L_q(\mathbb{R}^+)$ for $1 < q \le 2$.*

Proof. (1) For $1 \le p \le 2$, $z \in \mathbb{C}^+$ and $f \in L_p(\mathbb{R}^+)$ one has

$$[\mathcal{L}(\mathcal{C}_\nu^* f)](z) = \int_0^\infty \varphi_{\nu,p}(t) (\mathcal{L} \circ T_p(-t)) f(z) dt = \int_0^\infty \varphi_{\nu,p}(t) (T_q(t) \circ \mathcal{L}) f(z) dt = [\mathfrak{C}_\nu(\mathcal{L} f)](z),$$

where we have used (1.5) in the second equality. This gives us part (1). Part (2) is shown analogously. The proof is over. □

1.3 Range spaces of Cesàro-Hardy operators

The optimal domain and optimal range of distinguished operators have been discussed in recent times. The case of optimal domain of the classical Hardy operator is discussed in [DS]. Here we are interested in the range of operators $\mathcal{C}_\nu^*$ and $\mathcal{C}_\nu$ when the domains are L_p spaces as above. To begin with, recall that $\mathcal{C}_\nu^*\colon L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+)$, $1 \le p < \infty$, is injective. We give here a proof of this for the convenience of readers.

Let p be such that $1 \le p < \infty$ and take $f \in L_p(\mathbb{R}^+)$ such that $\mathcal{C}_\nu^* f = 0$. Then, for every $t > 0$,

$$\mathcal{C}_\nu^* f(t) = \nu \int_t^\infty \frac{(s-t)^{\nu-1}}{s^\nu} f(s)ds = \nu t^{\nu-1} \int_0^{1/t} (t^{-1}-r)^{\nu-1} r^{-1} f(r^{-1}) dr$$

Therefore $\mathcal{C}_\nu^* f = 0 \Leftrightarrow ((\cdot)_+^{\nu-1} * g) = 0$ where $g(r) := r^{-1}f(r^{-1})$. In addition, $g \in L^1_{loc}(\mathbb{R}^+)$ because $\int_0^A g(r)dr = \int_{1/A}^\infty \frac{f(t)}{t} dt$ for every $A > 0$. By Tichmarsh's convolution theorem [D1, p.188], one has $g = 0$ and so $f = 0$. (The injectivity of $\mathcal{C}_\nu^*$ is also a consequence of the density of the range of $\mathcal{C}_\nu$ on L_q spaces but we have preferred to appeal to the convolution direct argument.)

Then we define $\mathcal{T}_p^{(\nu)}(t^\nu) := \mathcal{C}_\nu^*(L_p(\mathbb{R}^+))$ endowed with the norm

$$\|f\|_{p,(\nu)} := \|(\mathcal{C}_\nu^*)^{-1}\|_p, \tag{1.9}$$

so that $\mathcal{T}_p^{(\nu)}(t^\nu)$ is a Banach space and $\mathcal{C}_\nu^*\colon L_p(\mathbb{R}^+) \to \mathcal{T}_p^{(\nu)}(t^\nu)$ is an (onto) isometry. Notice that $\mathcal{T}_p^{(0)}(t^0) = L_p(\mathbb{R}^+)$.

Spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$, $\nu \ge 0$ -and therefore the Cesàro-Hardy operators $\mathcal{C}_\nu^*$, $\mathbb{C}_\nu$- are intimately related with fractional derivatives and integrals:

Let $L_p(\mathbb{R}^+, t^{\nu p})$ denote the Banach space of measurable functions f such that $t \mapsto t^\nu f(t)$ belongs to $L_p(\mathbb{R}^+)$, and let τ_ν denote the multiplication operator by the (weight) function $t \mapsto t^\nu$, $t > 0$. Put $\mu_{-\nu} := \Gamma(\nu+1)\tau_{-\nu}$.

Set $W^\nu\colon \mathcal{T}_p^{(\nu)}(t^\nu) \xrightarrow{(\mathcal{C}_\nu^*)^{-1}} L_p(\mathbb{R}^+) \xrightarrow{\mu_{-\nu}} L_p(\mathbb{R}^+, t^{\nu p})$; that is,

$$W^\nu f(t) := \Gamma(\nu+1)t^{-\nu}\left[(\mathcal{C}_\nu^*)^{-1} f\right](t), \quad f \in \mathcal{T}_p^{(\nu)}(t^\nu),\ t > 0.$$

Just to clarify, we present this relation as a commutative diagram:

$$\begin{tikzcd}
\mathcal{T}_p^{(\nu)}(t^\nu) \arrow[r, "(\mathcal{C}_\nu^*)^{-1}"] \arrow[d, "W^\nu"'] & L_p(\mathbb{R}^+) \arrow[d, "\mu_\nu"] \\
L_p(\mathbb{R}^+, t^{\nu p}) \arrow[r, shift left, "\equiv"] & L_p(\mathbb{R}^+, t^{\nu p}) \arrow[l, shift left]
\end{tikzcd}$$

It is clear that W^ν has the inverse

$$W^{-\nu} g(t) := \int_t^\infty (s-t)^{\nu-1} g(s) \frac{ds}{\Gamma(\nu)}, \quad g \in L_p(\mathbb{R}^+, t^{\nu p}).$$

That is

$$\begin{array}{ccc} \mathcal{T}_p^{(\nu)}(t^\nu) & \xleftarrow{\ \mathcal{C}_\nu^*\ } & L_p(\mathbb{R}^+) \\ {\scriptstyle W^{-\nu}}\big\uparrow & & \big\uparrow{\scriptstyle \Gamma(\nu+1)^{-1}\tau_\nu} \\ L_p(\mathbb{R}^+, t^{\nu p}) & \underset{\longleftarrow}{\overset{\equiv}{\longrightarrow}} & L_p(\mathbb{R}^+, t^{\nu p}) \end{array}$$

Thus $\mathcal{T}_p^{(\nu)}(t^\nu)$ is formed by all the elements f in $L_p(\mathbb{R}^+)$ for which there exists a unique element in $L_p(\mathbb{R}^+, t^{\nu p})$, notated $W^\nu f$, such that

$$f(t) := \int_t^\infty (s-t)^{\nu-1} W^\nu f(s) \frac{ds}{\Gamma(\nu)}, \tag{1.10}$$

and so that

$$\|f\|_{p,(\nu)} := \left(\int_0^\infty |W^\nu f(t) t^\nu|^p dx \right)^{1/p} < \infty. \tag{1.11}$$

The operators W^ν and $W^{-\nu}$ introduced above are extensions of the corresponding restricted operators $W^\nu \colon \mathcal{S}_+ \to \mathcal{S}_+$, $W^{-\nu} \colon \mathcal{S}_+ \to \mathcal{S}_+$ which can be found in [SKM], [MR], for instance, as operators defining particular cases of fractional derivation and integration, respectively. The following properties emphasize the derivation character of W^ν. Set $W^0 := Id$, the identity operator, and $h_\lambda(t) = h(\lambda t)$, for any function h and $\lambda, t > 0$.

Proposition 1.3.1. (1) Integro-differentiation group property: $W^\nu \circ W^\mu = W^{\nu+\mu}$ *on* $\mathcal{S}_+$ *for every* $\nu, \mu \in \mathbb{R}$.

(2) $W^n \varphi = (-1)^n \varphi^{(n)}$, *for every* $\varphi \in \mathcal{S}_+$, *if* $\nu = n \in \mathbb{Z}$. *Hence, for every* $\nu > 0$ *and every integer* n *such that* $n \geq [\nu] + 1$,

$$W^\nu \varphi = (-1)^n \frac{d^n}{dt^n} W^{-(n-\nu)} \varphi.$$

(3) Homogeneity: $W^\nu f_\lambda = \lambda^\nu (W^\nu f)_\lambda$, *where* $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$.

Proof. (1) See [SKM, p. 96].

(2) For a negative integer n, the first equality is the formula of $-n$ times integration by parts. For positive n, the equality follows from the equality $W^n = (W^{-n})^{-1}$.

(3) This equality is straightforward. □

In fact, for later considerations, it is suitable to regard spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$ as being formed by "derivatives" of functions. Under this viewpoint, these spaces were introduced in [Mi1], [GM] (and previously in [AK] in the case $\nu \in \mathbb{N}$) in the setting of integrated semigroups and distribution semigroups (these families of "semigroups" are of interest to deal with ill-posed abstract Cauchy problems, see [ABHN]). Next, we list some of their properties.

Proposition 1.3.2. *Let $1 \le p < \infty$.*

(1) *$C_c^{(\infty)}(0,\infty)$ is dense in $\mathcal{T}_p^{(\nu)}(t^\nu)$ for all $\nu \ge 0$.*

(2) *For every $\nu > 0$, $\mathcal{T}_p^{(\nu)}(t^\nu)$ is a convolution Banach $\mathcal{T}_1^{(\nu)}(t^\nu)$-module; that is, there exists a constant $C_{p,\nu}$ such that for every $g \in \mathcal{T}_1^{(\nu)}(t^\nu)$ and $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$,*

$$g * f \in \mathcal{T}_p^{(\nu)}(t^\nu) \text{ with } \|g * f\|_{p,(\nu)} \le C_{p,\nu}\|g\|_{1,(\nu)}\|f\|_{p,(\nu)}.$$

Moreover, $\mathcal{T}_1^{(\nu)}(t^\nu) * \mathcal{T}_p^{(\nu)}(t^\nu)$ is dense in $\mathcal{T}_p^{(\nu)}(t^\nu)$.

(3) *For every μ, ν such that $\mu > \nu \ge 0$,*

$$\mathcal{T}_p^{(\mu)}(t^\mu) \hookrightarrow \mathcal{T}_p^{(\nu)}(t^\nu) \hookrightarrow L_p(\mathbb{R}^+).$$

(the hook arrows mean continuous inclusions).

Furthermore, for every $f \in \mathcal{T}_p^{(\mu)}(t^\mu)$,

$$W^\nu f(t) = \frac{1}{\Gamma(\mu-\nu)} \int_t^\infty (s-t)^{\mu-\nu-1} W^\mu f(s)\,ds, \quad t > 0, \tag{1.12}$$

whence

$$\sup_{t>0}\ t^{\nu+(1/p)}|W^\nu f(t)| \le C_{p,\nu,\mu}\|f\|_{p,(\mu)}, \qquad \text{provided } \mu > \nu + \frac{1}{p}, \tag{1.13}$$

for some constant $C_{p,\nu,\mu} > 0$.

(4) *For every $\nu > 0$,*

$$\begin{aligned} \mathcal{T}_p^{(\nu)}(t^\nu) &= \{f \in L_p(\mathbb{R}^+) \ :\ t^\nu W^\nu f \in L_p(\mathbb{R}^+)\} \\ &= \{f \in L_p(\mathbb{R}^+) \ :\ t^\mu W^\mu f \in L_p(\mathbb{R}^+),\ \forall\ 0 \le \mu \le \nu\} \end{aligned}$$

and the following norms are equivalent on $\mathcal{T}_p^{(\nu)}(t^\nu)$:

$$(1)\ \|f\|_{p,(\nu)} := \|t^\nu W^\nu f\|_p,$$

$$(2)\ \sup_{0\le\mu<\nu} \|f\|_{p,(\mu)} = \sup_{0\le\mu<\nu} \|t^\mu W^\mu f\|_p.$$

(5) *If $1 < p < \infty$ and q is the conjugate exponent of p, then the dual of $\mathcal{T}_p^{(\nu)}(t^\nu)$ can be represented by $\mathcal{T}_q^{(\nu)}(t^\nu)$, with duality given by*

$$\langle f, g\rangle_\nu \ = \ \frac{1}{\Gamma(\nu+1)^2} \int_0^\infty W^\nu f(t) W^\nu g(t) t^{2\nu}\,dt = \int_0^\infty (C_\nu^*)^{-1} f(t) (C_\nu^*)^{-1} g(t)\,dt.$$

for $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$, $g \in \mathcal{T}_q^{(\nu)}(t^\nu)$.

Proof. To begin with, we do observe two facts. First, $C_c^{(\infty)}[0,\infty)$ is dense in every $\mathcal{T}_p^{(\nu)}(t^\nu)$ for all $\nu > 0$ and $1 \le p < \infty$, since $\mathcal{C}_\nu^*\colon L_p(\mathbb{R}^+) \to \mathcal{T}_p^{(\nu)}(t^\nu)$ is a surjective isometry with $\mathcal{C}_\nu^*(C_c^{(\infty)}[0,\infty)) = C_c^{(\infty)}[0,\infty)$. On the other hand, take $\psi \in C_c^{(\infty)}(0,\infty)$ positive such that $\int_0^\infty \psi(t)\,dt = 1$. For $\varepsilon > 0$ put $\psi_\varepsilon(t) = \varepsilon^{-1}\psi(\varepsilon^{-1}t)$, $t \in \mathbb{R}^+$. Then $(\psi_\varepsilon)_{0<\varepsilon<1}$ is a bounded approximate identity for $\mathcal{T}_p^{(\nu)}(t^\nu)$ for every $\nu \ge 0$, that is, $\lim_{\varepsilon\to 0^+} f * \psi_\varepsilon = f$ in $\mathcal{T}_p^{(\nu)}(t^\nu)$ (see [GMR1, Proposition 2.3] for $p = 1$; for arbitrary p the argument is similar).

(1) Since $h * \psi_\varepsilon \in C_c^{(\infty)}(0,\infty)$ for every $h \in C_c^{(\infty)}[0,\infty)$ and $C_c^{(\infty)}[0,\infty)$ is dense in $\mathcal{T}_p^{(\nu)}(t^\nu)$, one gets that $C_c^{(\infty)}(0,\infty)$ is dense in $\mathcal{T}_p^{(\nu)}(t^\nu)$.

(2) Since $C_c^{(\infty)}[0,\infty)$ is dense in $\mathcal{T}_p^{(\nu)}(t^\nu)$ one can derive the module property of the statement as it is done in [R]. (A proof in a more general situation is given in Chapter 5 below, see Theorem 5.1.15).

(3) Let $\nu > 0$ and take μ such that $\mu > \nu \ge 0$. Let $f \in \mathcal{T}_p^{(\mu)}(t^\mu)$. The function ϕ given by the integral

$$\phi(t) := \int_t^\infty (s-t)^{\mu-\nu-1}|W^\mu f(s)|ds, \quad t > 0,$$

is an element of $L_p(t^{\nu p})$, so that that integral is finite for all $t > 0$ a. e. In effect,

$$\begin{aligned}\|\phi\|_{L_p(t^{\nu p})} &\le \left(\int_0^\infty \left[\int_t^\infty \frac{(s-t)^{\mu-\nu-1}}{s^{\mu-\nu}} s^\mu |W^\mu f(s)|ds\right]^p dt\right)^{1/p} \\ &\le C_{\nu,\mu}\|s^\mu W^\mu f(s)\|_p \;=\; C_{\nu,\mu}\|f\|_{p,(\mu)} \;<\; \infty,\end{aligned}$$

for some constant $C_{\nu,\mu} > 0$, where the second inequality is Hardy's inequality (4).

Hence, the function g given by

$$g(t) := \frac{1}{\Gamma(\mu-\nu)}\int_t^\infty (s-t)^{\mu-\nu-1} W^\mu f(s)\; ds,$$

is defined for a.e. $t > 0$ and $g \in L_p(t^{\nu p})$ with $\|g\|_{L_p(t^{\nu p})} \le D_{\nu,\mu}\|f\|_{p,(\mu)}$, for some constant $D_{\nu,\mu} > 0$.

Using the same argument as before (with ϕ instead $W^\mu f$ and ν in the exponent instead $\mu-\nu-1$), one gets $\int_t^\infty (s-t)^{\nu-1}\phi(s)ds < \infty$. Then by Fubini's theorem and (1.10), for every $t > 0$,

$$\begin{aligned}\frac{1}{\Gamma(\nu)}\int_t^\infty (s-t)^{\nu-1} g(s)ds &= \frac{1}{\Gamma(\nu)\Gamma(\mu-\nu)}\int_t^\infty\int_t^r (s-t)^{\nu-1}(r-s)^{\mu-\nu-1} ds W^\nu f(r)dr \\ &= \frac{B(\nu,\mu-\nu)}{\Gamma(\nu)\Gamma(\mu-\nu)}\int_t^\infty (r-t)^{\mu-1} W^\mu f(r)dr \\ &= \frac{1}{\Gamma(\mu)}\int_t^\infty (r-t)^{\mu-1} W^\mu f(r)dr \;=\; f(t).\end{aligned}$$

Therefore, applying the uniqueness of the representation (1.10) we obtain that $g = W^\mu f$. In other words, we have proved (1.12) and the continuous inclusion $\mathcal{T}_p^{(\mu)}(t^\mu) \hookrightarrow \mathcal{T}_p^{(\nu)}(t^\nu)$.

Finally, take $\mu > \nu + (1/p)$ and $f \in \mathcal{T}_p^{(\mu)}(t^\mu)$. Assume $1 < p < \infty$. For $t > 0$,

$$\begin{aligned}
|W^\nu f(t)| &\leq \frac{1}{\Gamma(\mu-\nu)} \int_t^\infty \frac{(s-t)^{\mu-\nu-1}}{s^\mu} s^\mu |W^\mu f(t)| ds \\
&\leq \frac{1}{\Gamma(\mu-\nu)} \left(\int_t^\infty (s-t)^{q(\mu-\nu-1)} s^{-q\mu} ds \right)^{1/q} \|f\|_{p,(\mu)} \\
&= \frac{B(q(\nu+1)-1, q(\mu-\nu-1)+1)^{1/q}}{\Gamma(\mu-\nu)} t^{-(\nu+\frac{1}{p})} \|f\|_{p,(\mu)}
\end{aligned}$$

Assume now $p = 1$. Then, for $t > 0$,

$$|W^\nu f(t)| \leq \frac{1}{\Gamma(\mu-\nu)} \left(\sup_{s>t} \frac{(s-t)^{\mu-\nu-1}}{s^\beta} \right) \|f\|_{1,(\mu)} = C_{\nu,\mu} t^{-(\nu+1)} \|f\|_{1,(\mu)}.$$

(4) This point follows readily from the definition of $\mathcal{T}_p^{(\nu)}(t^\nu)$, equality (1.12) and Hardy's inequality (4).

(5) It follows from $L_q(\mathbb{R}^+) \cong L_p(\mathbb{R}^+)'$ and the fact that $f \mapsto t^\nu W^\nu f$ is an isometry.

All in all, the proof is over. $\square$

We include here how operator W^ν acts on Riesz kernels $R_s^\theta(t)$:

$$W^\nu R_s^\theta(t) = R_s^{\theta-\nu}(t) \tag{1.14}$$

for $\nu > 0$, and integrable whenever $\theta > \nu - 1$ (see [GP, p.319]). In the limit case, $W^\nu R_s^{\nu-1}(t) = \delta_s(t)$.

We give in the next lemma some functions that belong (or not) to the spaces. We will use these examples later in this work.

Lemma 1.3.3. *If $\nu, a > 0$ and $p \geq 1$, then*

(1) $t^\mu \notin \mathcal{T}_p^{(\nu)}(t^\nu)$ *for* $\mu \in \mathbb{C}$.

(2) $(a+t)^{-\mu} \in \mathcal{T}_p^{(\nu)}(t^\nu)$ *for* $\mathrm{Re}\mu > 1/p$.

Proof. (1) It suffices to note that t^μ does not belong to $L_p(\mathbb{R}^+)$.

(2) For $0 < \mathrm{Re}\gamma < \mathrm{Re}\delta$ and $a > 0$ it is well known that $W^{-\gamma}(a+t)^{-\delta} = \frac{\Gamma(\delta-\gamma)}{\Gamma(\delta)}(t+a)^{\gamma-\delta}$, see for example [EMOT, p. 201]. With this formula, it is easy to check that

$$W^\nu (a+t)^{-\mu} = \frac{\Gamma(\nu+\mu)}{\Gamma(\mu)} (t+a)^{-(\nu+\mu)}.$$

Thus for $f(t) := (a+t)^{-\mu}$ we obtain

$$\begin{aligned}
||f||_{p,(\nu)}^p &= \frac{1}{\Gamma(\nu+1)^p} \int_0^\infty |W^\nu f(t)|^p t^{\nu p} dt = \left(\frac{\Gamma(\nu+\mu)}{\Gamma(\nu)\Gamma(\mu)} \right)^p \int_0^\infty \frac{t^{\nu p}}{|(t+a)^{(\nu+\mu)p}|} dt \\
&\leq \left(\frac{\Gamma(\nu+\mu)}{\Gamma(\nu)\Gamma(\mu)} \right)^p \int_0^\infty \frac{1}{(t+a)^{p \,\mathrm{Re}\,\mu}} dt < \infty,
\end{aligned}$$

and we conclude the proof. $\square$

Remark 1.3.4. For $n \in \mathbb{N}$, it follows from Proposition 1.3.1 and (1.13) that every function f in $\mathcal{T}_p^{(n)}(t^n)$ is $(n-1)$-times differentiable with $f^{(n-1)}$ absolutely continuous on $\mathbb{R}^+$ and such that $\int_0^\infty |f^{(n)}(t)|^p dt < \infty$. This suggests us to refer to $\mathcal{T}_p^{(\nu)}(t^\nu)$ as space of absolutely continuous functions of fractional order, when ν is any positive number. In order to distinguish the class $\mathcal{T}_p^{(\nu)}(t^\nu)$, $\nu > 0, 1 \le p < \infty$, from other classes of Sobolev type existing in the literature, each $\mathcal{T}_p^{(\nu)}(t^\nu)$ will be called Lebesgue-Sobolev space here.

Part (2) of Proposition 1.3.2 says in particular that $\mathcal{T}_1^{(\nu)}(t^\nu)$ is a Banach algebra. In this respect, $\mathcal{T}_1^{(\nu)}(t^\nu)$ has been studied in a number of papers (see [GMR1], [GMR2], [GMSt], [GS], for example). It sounds sensible to also study the structure of spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$, $1 < p < \infty$. From (1.13), it follows that $|f(t)| \le C\|f\|_{p,(\nu)} t^{-1/p}$, for every $\nu > 1/p$, $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$ and $t > 0$. This is to say that point evaluations are continuous on $\mathcal{T}_p^{(\nu)}(t^\nu)$. When $p = 2$ it means that spaces $\mathcal{T}_2^{(\nu)}(t^\nu)$ are Hilbert spaces with reproducing kernel. We call them RKH-Sobolev spaces and show, in Chapter 4, some of their features.

Chapter 2
Cesàro-Hardy operators on Sobolev spaces

In this chapter, based on [LMPS], we prove that the operator $\mathcal{C}_\beta$ and its companion $\mathcal{C}_\beta^*$ are bounded and commute on the spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$, that we have defined on Chapter 1, and on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$. The space $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ is the whole real line version of $\mathcal{T}_p^{(\nu)}(t^\nu)$, see the final section of this chapter. Note that sometimes we use $\beta > 0$ as index of the operators (instead of ν) if there is no correlation between the operator and the space it is acting on.

2.1 Working on the half line $\mathbb{R}^+$

2.1.1 Composition group on Sobolev spaces defined on $\mathbb{R}^+$

First of all, the reader should notice that the group $(T_p(t))_{t\in\mathbb{R}}$ considered in Chapter 1 acting on $L_p(\mathbb{R}^+)$ also acts on $\mathcal{T}_p^{(\nu)}(t^\nu)$ since this space is a subspace of $L_p(\mathbb{R}^+)$, see Proposition 1.3.2 (3). In fact, $T_p(t)$, $t \in \mathbb{R}$, are isometries from $\mathcal{T}_p^{(\nu)}(t^\nu)$ onto itself:

$$\begin{aligned}\|T_p(t)f\|_{p,(\nu)}^p &= \int_0^\infty |W^\nu T_p(t)f(s)|^p s^{\nu p} ds \;=\; e^{-t}\int_0^\infty |W^\nu f(e^{-t}s)|^p s^{\nu p} ds \\ &= e^{-t}\int_0^\infty e^t |e^{-\nu t}(W^\nu f)(u)|^p (e^{\nu t}u^\nu)^p du \;=\; \|f\|_{p,(\nu)}^p.\end{aligned}$$

Next we do observe that the group commutes with the operator $(\mathcal{C}_\nu^*)^{-1}$ on $\mathcal{T}_p^{(\nu)}(t^\nu)$.

Lemma 2.1.1. *Let $p \geq 1$, $\nu \geq 0$ and $t \in \mathbb{R}$. Then $(\mathcal{C}_\nu^*)^{-1} \circ T_p(t) = T_p(t) \circ (\mathcal{C}_\nu^*)^{-1}$.*

Proof. It is Corollary 1.2.2 (1) simply by reversing the diagram

$$\begin{array}{ccc} L_p(\mathbb{R}^+) & \xrightarrow{\mathcal{C}_\nu^*} & \mathcal{T}_p^\nu(t^\nu) \\ {\scriptstyle T_p(t)}\downarrow & & \downarrow{\scriptstyle T_p(t)} \\ L_p(\mathbb{R}^+) & \xrightarrow{\mathcal{C}_\nu^*} & \mathcal{T}_p^\nu(t^\nu) \end{array}$$

□

The following result will be the key in the study of spectral properties of the generalized Cesàro operators $\mathcal{C}_\beta$ and $\mathcal{C}_\beta^*$ defined on Sobolev spaces; it is the generalization of the properties of the group defined in (1.1).

Theorem 2.1.2. *For $p \geq 1$ and $\nu \geq 0$, the family of operators $(T_p(t))_{t\in\mathbb{R}}$ defined by*

$$T_p(t)f(s) := e^{-t/p} f(e^{-t}s), \qquad a.e\ s > 0, \quad f \in \mathcal{T}_p^{(\nu)}(t^\nu),$$

is a C_0-group of isometries on $\mathcal{T}_p^{(\nu)}(t^\nu)$ whose infinitesimal generator Λ_p is given by

$$(\Lambda_p f)(s) := -sf'(s) - \frac{1}{p} f(s)$$

with domain $D(\Lambda_p) = \mathcal{T}_p^{(\nu+1)}(t^{\nu+1})$.

Proof. We know that the family $(T_p(t))_{t\in\mathbb{R}}$ is a group of operators, and we have already checked that $T_p(t)$ are isometries on $\mathcal{T}_p^{(\nu)}(t^\nu)$. To prove that they are strongly continuous, we use Lemma 2.1.1: Let $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$. Then,

$$\|T_p(t)f - f\|_{(\nu),p} = \|(\mathcal{C}_\nu^*)^{-1}(T_p(t)f - f)\|_p = \|T_p(t)((\mathcal{C}_\nu^*)^{-1}f) - (\mathcal{C}_\nu^*)^{-1}f\|_p \to 0 \quad (\text{as } t \to 0)$$

because we know that $(T_p(t))_{t\in\mathbb{R}}$ are strongly continuous on $L_p(\mathbb{R}^+)$.

On $\mathcal{T}_p^{(\nu)}(t^\nu)$ define $(S_t)_{t\geq 0}$ by $S_t(f)(s) := f(e^{-t}s)$. Then, an easy computation shows that the generator A of $(S_t)_{t\geq 0}$ with domain $\{f \in \mathcal{T}_p^{(\nu)}(t^\nu) \ : \ tf' \in \mathcal{T}_p^{(\nu)}(t^\nu)\}$ is given by $Af(s) = -sf'(s)$. Therefore, the rescaled semigroup $(T_p(t))_{t\geq 0}$ has domain $\{f \in \mathcal{T}_p^{(\nu)}(t^\nu) \ : \ tf' \in \mathcal{T}_p^{(\nu)}(t^\nu)\}$ and its generator is $(\Lambda_p f)(s) = -sf'(s) - \frac{1}{p}f(s)$. See [EN, p. 60] for more details.

Finally, we prove that $D(\Lambda_p) = \mathcal{T}_p^{(\nu+1)}(t^{\nu+1})$. In fact, let $f \in \mathcal{T}_p^{(\nu+1)}(t^{\nu+1})$ be given. Since $\mathcal{T}_p^{(\nu+1)}(t^{\nu+1}) \hookrightarrow \mathcal{T}_p^{(\nu)}(t^\nu)$, we have $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$. From [MR, p. 246] it is easy to show that $W^\nu(tf'(t)) = \nu W^\nu f(t) + tW^{\nu+1}f(t)$. Thus, $tf' \in \mathcal{T}_p^{(\nu)}(t^\nu)$ and therefore $f \in D(\Lambda_p)$. Conversely, if $f \in D(\Lambda_p)$, then $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$ and $tf' \in \mathcal{T}_p^{(\nu)}(t^\nu)$. The same identity as above reads $t^{\nu+1}W^{\nu+1}f(t) = t^\nu W^\nu(tf'(t)) - \nu t^\nu W^\nu f(t)$, and therefore $f \in \mathcal{T}_p^{(\nu+1)}(t^{\nu+1})$. □

The proof of the following result is inspired in [AS, Proposition 2.3] (see also [Ar]).

Proposition 2.1.3. *For $1 \leq p < \infty$ we have*

(1) $\sigma_\pi(\Lambda_p) = \varnothing$;

(2) $\sigma(\Lambda_p) = i\mathbb{R}$.

Proof. (1) Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$ such that $\Lambda_p(f) = \lambda f$. Then, f is solution of the differential equation

$$sf'(s) + \left(\lambda + \frac{1}{p}\right) f(s) = 0.$$

The nonzero solutions to this equation have the form $f(t) = ct^{-(\lambda+1/p)}$ with $c \neq 0$. But clearly, by Lemma 1.3.3, these solutions are not in $\mathcal{T}_p^{(\nu)}(t^\nu)$. Therefore $\sigma_\pi(\Lambda_p) = \varnothing$.

(2) Since each $T_p(t)$ is an invertible isometry its spectrum satisfies

$$\sigma(T_p(t)) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

By the spectral mapping theorem (see [EN, Theorem IV.3.6]), we have that

$$e^{t\sigma(\Lambda)} \subseteq \sigma(T_p(t)).$$

Therefore, if $w \in \sigma(\Lambda_p)$, then $e^{tw} \in \{z \in \mathbb{C} : |z| = 1\}$. Thus, we obtain that $\sigma(\Lambda_p) \subseteq i\mathbb{R}$.

Conversely, let $\mu \in i\mathbb{R}$ and assume that $\mu \in \rho(\Lambda_p)$. Let $\lambda = \mu + \frac{1}{p}$. By Lemma 1.3.3 the function f defined by $f(t) := (1+t)^{-\lambda-1} \in \mathcal{T}_p^{(\nu)}(t^\nu)$. Since $R(\mu, \Lambda_p)$ is a bounded operator, the function $g := R(\mu, \Lambda_p)f$ belongs to $\mathcal{T}_p^{(\nu)}(t^\nu)$. Therefore, g is solution of the equation

$$\lambda g(t) + tg'(t) = f(t), \qquad t > 0.$$

An easy computation shows that the solution of this equation is $G(t) := ct^{-\lambda} + \lambda^{-1}(1+t)^{-\lambda}$, where c is a constant. However, as in Lemma 1.3.3 one can check that $G \notin \mathcal{T}_p^{(\nu)}(t^\nu)$. Therefore, $\mu \in \sigma(\Lambda_p)$. □

Now, consider the negative part $(T_p(-t))_{t\geq 0}$ of the group $(T_p(t))_{t\in\mathbb{R}}$: that is, for $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$,

$$T_p(-t)f(s) = e^{t/p} f(e^t s), \quad t \geq 0.$$

Obviously, $(T_p(-t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{T}_p^{(\nu)}(t^\nu)$ of isometries whose generator is $-\Lambda_p$.

We finish this section establishing the relationship between the semigroups $(T_p(t))_{t\geq 0}$ and $(T_q(-t))_{t\geq 0}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 2.1.4. *The semigroups $(T_{t,p})_{t\geq 0}$ and $(T_{-t,q})_{t\geq 0}$ are dual operators of each other acting on $\mathcal{T}_p^{(\nu)}(t^\nu)$ and $\mathcal{T}_q^{(\nu)}(t^\nu)$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. This is easily checked by Proposition 1.3.2 (5) and Proposition 1.3.1 (3). □

2.1.2 Cesàro-Hardy operators on Sobolev spaces defined on $\mathbb{R}^+$

For $\beta > 0$, we can consider the generalized Cesàro-Hardy operator $\mathcal{C}_\beta$ restricted to $\mathcal{T}_p^{(\nu)}(t^\nu)$ for any $\nu > 0$. Recall its expression

$$\mathcal{C}_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s) ds = \int_0^\infty \varphi_{\beta,q}(r) T_p(r) f(t) dr, \quad t > 0.$$

(Note that we use β as subindex of the operator just to emphasize that there is no correlation with the order ν of the space). This formula is the result of applying Proposition 1.2.1 to the restriction of $T_p(t)$ to the space $\mathcal{T}_p^{\nu}(t^\nu)$ (applying also Theorem 2.1.2). Note that by this equality $\mathcal{C}_\beta$ is well defined and is a bounded operator on $\mathcal{T}_p^{(\nu)}(t^\nu)$ for $p > 1$.

With the function $\mathfrak{r}_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $(t > 0)$, we obtain the also equivalent formulation of the generalized Cesàro operator in terms of finite convolution:

$$\mathcal{C}_\beta f(t) := \frac{1}{\mathfrak{r}_{\beta+1}(t)} \int_0^t \mathfrak{r}_\beta(t-s) f(s) ds = \frac{1}{\mathfrak{r}_{\beta+1}(t)} (\mathfrak{r}_\beta * f)(t), \quad t > 0.$$

We remark that for certain classes of vector-valued functions f, the asymptotic behavior as $t \to \infty$ of $\mathcal{C}_\beta f(t)$ in the above representation has been studied in [LP].

Now we calculate $\mathcal{C}_\beta f$ for some particular functions:

Examples 2.1.5. **(1)** Functions $\mathfrak{r}_\gamma$ are eigenfunctions of $\mathcal{C}_\beta$ (although they do not belong to $L_p(\mathbb{R}^+)$), with eigenvalues $\frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$ respectively:

$$\mathcal{C}_\beta(\mathfrak{r}_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)t^{\beta-1}} \int_0^t (t-s)^{\beta-1} s^{\gamma-1} ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \mathfrak{r}_\gamma(t), \qquad t > 0.$$

(2) Take $e_\lambda(t) := e^{-\lambda t}$ for $t > 0$ and $\lambda \in \mathbb{C}^+$. Then

$$\mathcal{C}_1(e_\lambda)(t) = \frac{1}{\lambda t}(1 - e^{-\lambda t}), \qquad \mathcal{C}_2(e_\lambda)(t) = \frac{2}{\lambda t}(e^{-\lambda t} - 1 + \lambda t), \qquad t > 0.$$

Since $\mathcal{C}_1^2(e_\lambda)(t) = \frac{1}{t\lambda} \int_0^t \frac{1 - e^{-\lambda s}}{s} ds$ for $t > 0$, we conclude that $\mathcal{C}_1^2(e_\lambda) \neq \mathcal{C}_2(e_\lambda)$ and then $\mathcal{C}_1^2 \neq \mathcal{C}_2$.

(3) More generally, take $f_\lambda(t) := E_{\beta,1}(\lambda t^\beta)$ the Mittag-Leffler function, for $t > 0$ and $\lambda \in \mathbb{C}^+$. Then

$$\mathcal{C}_\beta(f_\lambda)(t) = \frac{1}{\lambda \mathfrak{r}_{\beta+1}(t)}(1 - f_\lambda(t)), \quad t > 0.$$

The relationship between Cesàro-Hardy operators and fractional evolution equations of order ν can be seen in [LP].

The next lemma shows a commutativity property.

Lemma 2.1.6. *Take $\nu \geq 0$, $p > 1$ and $\beta > 0$. Then*

$$\mathcal{C}_\beta \circ (\mathcal{C}_\nu^*)^{-1} = (\mathcal{C}_\nu^*)^{-1} \circ \mathcal{C}_\beta, \qquad \text{on } \mathcal{T}_p^{(\nu)}(t^\nu).$$

Proof. This is Corollary 1.2.2 (2), since $\mathcal{C}_\beta(\mathcal{T}_p^{(\nu)}(t^\nu)) \subseteq \mathcal{T}_p^{(\nu)}(t^\nu)$. □

Next, we calculate the norm of $\mathcal{C}_\beta$.

Theorem 2.1.7. *For $\nu \geq 0$, $p > 1$ and $\beta > 0$, if $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$ then*

$$\mathcal{C}_\beta f(t) = \int_0^\infty \varphi_{\beta,q}(r) T_p(r) f(t) dr, \quad t > 0, \tag{2.1}$$

where q is the conjugate exponent of p. The operator $\mathcal{C}_\beta$ is a bounded operator on $\mathcal{T}_p^{(\nu)}(t^\nu)$ and

$$\|\mathcal{C}_\beta\| = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)} = \beta B(\beta, 1-1/p).$$

Proof. Formula (2.1) is that one at the beginning of Subsection 2.1.2. Then, we have

$$\begin{aligned}\|\mathcal{C}_\beta f\|_{p,(\nu)} &\leq \int_0^\infty \varphi_{\beta,q}(r) \|T_p(r) f\|_{p,(\nu)} dr \;=\; \|f\|_{p,(\nu)} \int_0^\infty \varphi_{\beta,q}(r) dr \\ &= \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)} \|f\|_{p,(\nu)}.\end{aligned}$$

To check the exact value of $\|\mathcal{C}_\beta\|$, note that by Lemma 2.1.6, the boundedness of $\mathcal{C}_\beta$ on $L_p(\mathbb{R}^+)$ and the fact that the operator $(\mathcal{C}_\nu^*)^{-1}$ is an isometry, we have

$$\begin{aligned}\|\mathcal{C}_\beta\|_{p,(\nu)} &= \sup_{f\neq 0} \frac{\|\mathcal{C}_\beta f\|_{p,(\nu)}}{\|f\|_{p,(\nu)}} = \sup_{f \neq 0} \frac{\|(\mathcal{C}_\nu^*)^{-1} \circ \mathcal{C}_\beta f\|_p}{\|(\mathcal{C}_\nu^*)^{-1} f\|_p} \\ &= \sup_{f\neq 0} \frac{\|\mathcal{C}_\beta \circ (\mathcal{C}_\nu^*)^{-1} f\|_p}{\|(\mathcal{C}_\nu^*)^{-1} f\|_p} = \sup_{g\neq 0} \frac{\|\mathcal{C}_\beta g\|_p}{\|g\|_p} = \|\mathcal{C}_\beta\|_p.\end{aligned}$$

Finally, we observe that $\|\mathcal{C}_\beta\|_p = \inf\{M > 0 : \|\mathcal{C}_\beta f\|_p \leq M\|f\|_p\} = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}$ because, by (3), the constant $\frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}$ is optimal for the inequality. □

Remark 2.1.8. Let $p > 1$ be given. Take $\beta = 1$ and $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$. Then

$$\mathcal{C}_1 f(t) = \int_0^\infty e^{-r(1-1/p)} T_p(r) f(t) dr = R(\lambda_p, \Lambda) f(t), \quad \lambda_p = 1 - 1/p > 0. \tag{2.2}$$

and by the spectral theorem for resolvent operators (see for example [EN, Theorem IV.1.13]) we get that

$$\sigma(\mathcal{C}_1) = \left\{ w \in \mathbb{C} \;:\; \left| w - \frac{p}{2(p-1)} \right| = \frac{p}{2(p-1)} \right\}, \tag{2.3}$$

see [Mo1, Theorem 2] and similar results in [AS, Theorem 3.1] and [AP, Corollary 2.2]. Here, $R(\cdot, \Lambda)$ denotes the resolvent operator of Λ.

Note that in case $\beta = 2$ we obtain

$$\mathcal{C}_2 f(t) = 2\int_0^\infty e^{-r(1-1/p)}(1-e^{-r}) T_p(r) f(t) dr = 2R(\lambda_p, \Lambda) f(t) - 2R(\lambda_p + 1, \Lambda) f(t),$$

and, more generally, for $\beta = n + 1$,

$$\mathcal{C}_{n+1} f(t) = (n+1) \sum_{k=0}^{n} \binom{n}{k} (-1)^k R(\lambda_p + k, \Lambda) f(t), \quad n \in \mathbb{Z}^+.$$

In the next result, we are able to describe $\sigma(\mathcal{C}_\beta)$ for $\beta > 0$.

Theorem 2.1.9. *Let $1 < p < \infty$, $\nu, \beta > 0$ and $\mathcal{C}_\beta : \mathcal{T}_p^{(\nu)}(t^\nu) \to \mathcal{T}_p^{(\nu)}(t^\nu)$ the generalized Cesàro-Hardy operator. Then*

$$\sigma(\mathcal{C}_\beta) = \beta\overline{B(\beta, 1 - 1/p + i\mathbb{R})} := \Gamma(\beta+1)\overline{\left\{ \frac{\Gamma(1 - \frac{1}{p} + it)}{\Gamma(\beta + 1 - \frac{1}{p} + it)} \ : \ t \in \mathbb{R} \right\}}.$$

Proof. Note that $(T_p(t))_{t\in\mathbb{R}}$ is an uniformly bounded C_0-group (Theorem 2.1.2) whose infinitesimal generator is $(\Lambda_p, D(\Lambda_p))$ and $\mathcal{C}_\beta = \mathcal{F}f_{\beta,q}(\Lambda_p)$, i.e,

$$\mathcal{C}_\beta f = \int_0^\infty \varphi_{\beta,q}(r) T_p(r) f dr = \int_{-\infty}^\infty f_{\beta,q}(r) T_p(r) f dr,$$

where q is the conjugate exponent of p, $f_{\beta,q}(r) = \chi_{[0,\infty)}(r)\varphi_{\beta,q}(r)$ for $r \in \mathbb{R}$, see Theorem 2.1.7, and $\mathcal{F}$ is the Fourier transform. By [Sef, Theorem 3.1], we obtain

$$\sigma(\mathcal{C}_\beta) = \overline{\mathcal{F}f_{\beta,p}(\sigma(i\Lambda_p))}.$$

As $\sigma(i\Lambda) = \mathbb{R}$ (see Proposition 2.1.3 (2)) and $\mathcal{F}f_{\beta,p}(t) = \mathcal{L}f_{\beta,p}(it)$ we use that

$$\mathcal{L}f_{\beta,p}(z) = \beta \int_0^\infty e^{-zr}(1 - e^{-r})^{\beta-1} e^{-r(1-1/p)} dr = \frac{\Gamma(\beta+1)\Gamma(1 - \frac{1}{p} + z)}{\Gamma(\beta + 1 - \frac{1}{p} + z)}, \quad z \in \overline{\mathbb{C}^+}.$$

to conclude the result. $\square$

We include the picture of some particular cases of spectra just to illustrate how they are. (See Figures 2.1 and 2.2).

Remark 2.1.10. In the case that $n \in \mathbb{N}$, we obtain that

$$\sigma(\mathcal{C}_n) = \left\{ \frac{n!p^n}{((n+it)p - 1)\dots((1+it)p - 1)} \ : \ t \in \mathbb{R} \right\} \cup \{0\},$$

and for $n = 1$

$$\sigma(\mathcal{C}_1) = \left\{ \frac{p}{(1+it)p - 1} \ : \ t \in \mathbb{R} \right\} \cup \{0\} = \left\{ w \in \mathbb{C} \ : \ \left| w - \frac{p}{2(p-1)} \right| = \frac{p}{2(p-1)} \right\}.$$

Now we consider the generalized dual Cesàro operator $\mathcal{C}_\beta^*$ on $\mathcal{T}_p^{(\nu)}(t^\nu)$. Recall that it is defined by

$$\mathcal{C}_\beta^* f(t) := \beta \int_t^\infty \frac{(s-t)^{\beta-1}}{s^\beta} f(s) ds = \beta \int_1^\infty \frac{(r-1)^{\beta-1}}{r^\beta} f(tr) dr, \quad t > 0.$$

For $0 < \gamma < 1$, functions $\mathfrak{r}_\gamma$ are eigenfunctions of $\mathcal{C}_\beta^*$ with eigenvalue $\frac{\Gamma(\beta+1)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}$:

$$\mathcal{C}_\beta^*(\mathfrak{r}_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)} \int_t^\infty \frac{(s-t)^{\beta-1}s^{\gamma-1}}{s^\beta} ds = \frac{\Gamma(\beta+1)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)} \mathfrak{r}_\gamma(t),$$

for $t > 0$.

Of course, we have the analogous of Lemma 2.1.6 for $\mathcal{C}_\beta^*$.

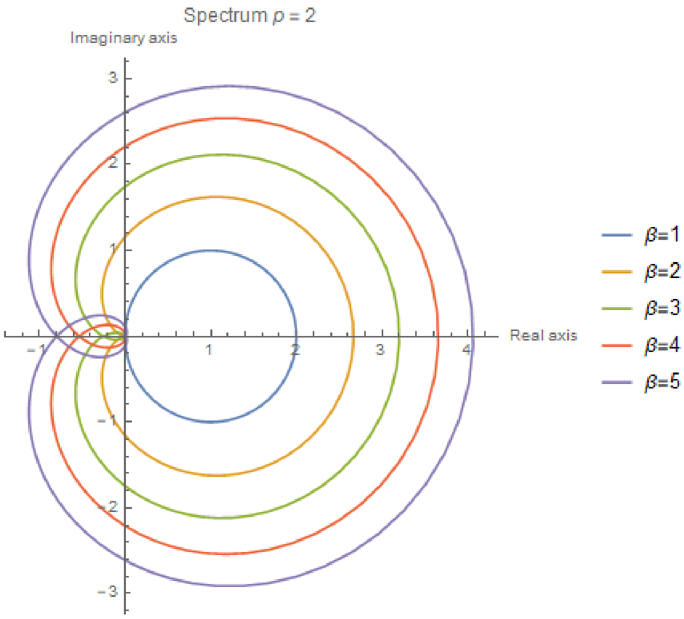


Figure 2.1: $\sigma(\mathcal{C}_\beta)$ for $p = 2$ and $\beta = 1, 2, 3, 4, 5$

Lemma 2.1.11. *Take $\nu \geq 0$, $p \geq 1$ and $\beta > 0$. Then*

$$\mathcal{C}^*_\beta \circ (\mathcal{C}^*_\nu)^{-1} = (\mathcal{C}^*_\nu)^{-1} \circ \mathcal{C}^*_\beta, \qquad \text{on } \mathcal{T}_p^{(\nu)}(t^\nu). \tag{2.4}$$

Hence the proof of the next result follows from duality and Theorem 2.1.7.

Theorem 2.1.12. *The operator $\mathcal{C}^*_\beta$ is a bounded operator on $\mathcal{T}_p^{(\nu)}(t^\nu)$ and*

$$\|\mathcal{C}^*_\beta\| = \frac{\Gamma(\beta+1)\Gamma(1/p)}{\Gamma(\beta+1/p)},$$

for $\nu \geq 0$, $p > 1$ and $\beta > 0$. The dual operator of $\mathcal{C}_\beta$ on $\mathcal{T}_p^{(\nu)}(t^\nu)$ is $\mathcal{C}^*_\beta$ on $\mathcal{T}_q^{(\nu)}(t^\nu)$, i.e.

$$\langle \mathcal{C}_\beta f, g\rangle_\nu = \langle f, \mathcal{C}^*_\beta g\rangle_\nu, \qquad f \in \mathcal{T}_p^{(\nu)}(t^\nu), \quad g \in \mathcal{T}_q^{(\nu)}(t^\nu),$$

where $\langle \quad , \quad \rangle_\nu$ is given in Proposition 1.3.2 (5) and $\frac{1}{p} + \frac{1}{q} = 1$.

If $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$, then

$$\mathcal{C}^*_\beta f(t) = \int_0^\infty \varphi_{\beta,p}(r) T_p(-r) f(t)\, dr, \quad t \geq 0. \tag{2.5}$$

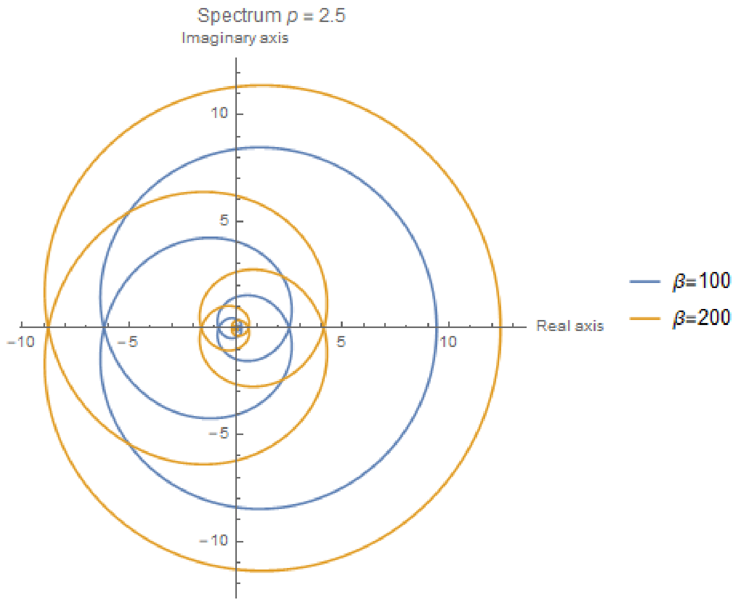


Figure 2.2: $\sigma(\mathcal{C}_\beta)$ for $p = 2.5$ and $\beta = 100, 200$

Remark 2.1.13. Take $\beta = 1$ and $f \in \mathcal{T}_p^{(\nu)}(t^\nu)$. Then

$$\mathcal{C}_1^* f(t) = \int_0^\infty e^{-r/p} T_p(-r) f(t) dr = R(1/p, -\Lambda_p) f(t), \qquad t \geq 0,$$

and by the spectral theorem for the resolvent operator, see [EN, Theorem IV.1.13], we obtain

$$\sigma(\mathcal{C}_1^*) = \left\{ w \in \mathbb{C} \ : \ \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\}.$$

This gives a proof of a conjecture posed by F. Móricz in [Mo1, Section 2]. See a similar result in [AS, Theorem 3.2].

In the following theorem we describe $\sigma(\mathcal{C}_\beta^*)$ for $\beta > 0$. The proof follows from duality and Theorem 2.1.9.

Theorem 2.1.14. *Let $\beta > 0$, $1 \leq p < \infty$, and $\mathcal{C}_\beta^* : \mathcal{T}_p^{(\nu)}(t^\nu) \to \mathcal{T}_p^{(\nu)}(t^\nu)$ the generalized dual Cesàro operator. Then*

$$\sigma(\mathcal{C}_\beta^*) = \beta \overline{B(\beta, 1/p + i\mathbb{R})} := \Gamma(\beta+1) \overline{\left\{ \frac{\Gamma(\frac{1}{p} + it)}{\Gamma(\beta + \frac{1}{p} + it)} \ : \ t \in \mathbb{R} \right\}}.$$

Remark 2.1.15. In the case that $n \in \mathbb{N}$, we obtain that

$$\sigma(\mathcal{C}_n^*) = \left\{ \frac{n!p^n}{((n-1)p+1+it)\dots(p+1+it)(1+it)} \ : \ t \in \mathbb{R} \right\} \cup \{0\},$$

and for $n = 1$

$$\sigma(\mathcal{C}_1^*) = \left\{ \frac{p}{1+it} \ : \ t \in \mathbb{R} \right\} \cup \{0\} = \left\{ w \in \mathbb{C} \ : \ \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\}.$$

Remark 2.1.16. In the case that $p = 2$ we have $\sigma(\mathcal{C}_\beta) = \sigma(\mathcal{C}_\beta^*)$ for all $\beta > 0$. Note that in case $p \neq 2$ the spectrum of $\mathcal{C}_\beta$ and $\mathcal{C}_\beta^*$ are dual in the sense that $\sigma(\mathcal{C}_\beta)$, with $\mathcal{C}_\beta$ defined on $\mathcal{T}_p^{(\nu)}(t^\nu)$, is identical to $\sigma(\mathcal{C}_\beta^*)$, with $\mathcal{C}_\beta^*$ defined on $\mathcal{T}_q^{(\nu)}(t^\nu)$, and where $\frac{1}{p} + \frac{1}{q} = 1$.

To finish this section we want to highlight the remarkable fact that $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta^*$ commute on $L_p(\mathbb{R}^+)$ (and then on $\mathcal{T}_p^{(\nu)}(t^\nu)$). We have already proved this commutativity on Corollary 1.2.2, we now give explicitly the value of $\mathcal{C}_\alpha \mathcal{C}_\beta^*$ in terms of the Gaussian hypergeometric function ${}_2F_1$. This theorem includes [Mo1, Lemma 2] for $\alpha = \beta = 1$.

Theorem 2.1.17. *Let $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta^*$ the generalized Cesàro operators on $L_p(\mathbb{R}^+)$ for $p > 1$. Then $\mathcal{C}_\alpha \mathcal{C}_\beta^* = \mathcal{C}_\beta^* \mathcal{C}_\alpha$ for $\alpha, \beta > 0$ and*

$$\begin{aligned}(\mathcal{C}_\alpha \mathcal{C}_\beta^*)f(t) = \alpha \int_0^t f(r) \frac{1}{t-r} \left(\frac{t-r}{t} \right)^{\alpha+\beta} {}_2F_1 \left(\alpha+\beta, \beta; \beta+1; \frac{r}{t} \right) dr \\ + \ \beta \int_t^\infty f(r) \frac{1}{r-t} \left(\frac{r-t}{t} \right)^{\alpha+\beta} {}_2F_1 \left(\alpha+\beta, \alpha; \alpha+1; \frac{t}{r} \right) dr,\end{aligned}$$

in particular

$$\begin{aligned}(\mathcal{C}_1 \mathcal{C}_\beta^*)f(t) &= \mathcal{C}_1 f(t) + \beta \int_t^\infty f(r) \frac{(r-t)^\beta}{r^{\beta+1}} {}_2F_1 \left(\beta+1, 1; 2; \frac{r}{t} \right) dr, \\ (\mathcal{C}_\alpha \mathcal{C}_1^*)f(t) &= \alpha \int_0^t f(r) \frac{(t-r)^\alpha}{t^{\alpha+1}} {}_2F_1 \left(\alpha+1, 1; 2; \frac{r}{t} \right) dr + \mathcal{C}_1^* f(t), \\ (\mathcal{C}_1 \mathcal{C}_1^*)f &= \mathcal{C}_1 f + \mathcal{C}_1^* f \ = \ (\mathcal{C}_1^* \mathcal{C}_1)f,\end{aligned}$$

for $f \in L_p(\mathbb{R}^+)$ and t almost everywhere on $\mathbb{R}^+$.

Proof. For $\alpha, \beta > 0$ and $f \in L_p(\mathbb{R}^+)$ and we apply the Fubini theorem to get

$$\begin{aligned}\mathcal{C}_\alpha \mathcal{C}_\beta^* f(t) = \mathcal{C}_\beta^* \mathcal{C}_\alpha f(t) &= \beta\alpha \int_t^\infty \frac{(x-t)^{\beta-1}}{x^{\beta+\alpha}} \int_0^x (x-r)^{\alpha-1} f(r) dr dx \\ &= \beta\alpha \int_0^\infty f(r) \int_{\max\{t,r\}}^\infty \frac{(x-t)^{\beta-1}(x-r)^{\alpha-1}}{x^{\beta+\alpha}} dx dr\end{aligned}$$

for t almost everywhere on $\mathbb{R}^+$. Note now that, for $0 < r < t$,

$$\int_t^\infty \frac{(x-t)^{\beta-1}(x-r)^{\alpha-1}}{x^{\beta+\alpha}} dx = \frac{1}{\beta(t-r)} \left(\frac{t-r}{t} \right)^{\alpha+\beta} {}_2F_1 \left(\alpha+\beta, \beta; \beta+1; \frac{r}{t} \right);$$

see for example [GR, p. 314, 3197(1)]. This proves the first formula of the statement.

Now take $\alpha = 1$. Since

$$(1-z)^a \, {}_2F_1(a,b;c;z) = {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

(see for example [MOS, p.47]), we get

$$\frac{1}{t-r}\left(\frac{t-r}{t}\right)^{1+\beta} {}_2F_1\left(1+\beta, \beta; \beta+1; \frac{r}{t}\right) = \frac{1}{t-r}\, {}_2F_1\left(1+\beta, 1; 1+\beta; \frac{-r}{t-r}\right) = \frac{1}{t}$$

where we apply that ${}_2F_1(-a,b;b;-z) = (1+z)^a$ ([MOS, p. 38]). The case $\beta = 1$ is proven similarly. $\square$

2.2 Extension to the whole line $\mathbb{R}$

2.2.1 Lebesgue-Sobolev spaces on $\mathbb{R}$

In this section we introduce the subspaces $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ which are contained in $L_p(\mathbb{R})$, similarly to $\mathcal{T}_p^{(\nu)}(t^\nu)$ are in $L_p(\mathbb{R}^+)$. We have not defined these spaces in the first chapter because we work in $\mathbb{R}$ just in this section.

First of all, we introduce Cesàro-Hardy operators for functions on the whole real line $\mathbb{R}$. For any function $g : \mathbb{R} \to \mathbb{C}$, set $g_- := g\chi_{(-\infty,0)}$, $g_+ := g\chi_{(0,\infty)}$. Let $1 \le p < \infty$ and let $f \in L_p(\mathbb{R})$. We define

$$\text{(2.6)} \qquad \mathcal{C}^*_{\nu,\mathbb{R}} f(t) = \begin{cases} \mathcal{C}^*_\nu f_-(t), & \text{if } t < 0, \\ \mathcal{C}^*_\nu f_+(t), & \text{if } t > 0. \end{cases} = \begin{cases} \nu \displaystyle\int_{-\infty}^t \frac{(t-s)^{\nu-1}}{(-s)^\nu} f(s)ds, & \text{if } t<0, \\ \nu \displaystyle\int_t^\infty \frac{(s-t)^{\nu-1}}{s^\nu} f(s)ds, & \text{if } t>0. \end{cases}$$

and for $q > 1$ and $g \in L_p(\mathbb{R})$,

$$\text{(2.7)} \quad \mathcal{C}_{\nu,\mathbb{R}} g(t) = \begin{cases} \mathcal{C}_\nu g_-(t), & \text{if } t < 0, \\ \mathcal{C}_\nu g_+(t), & \text{if } t > 0. \end{cases} = \begin{cases} \dfrac{\nu}{(-t)^\nu} \displaystyle\int_t^0 (s-t)^{\nu-1} g(s)ds, & \text{if } t<0, \\ \dfrac{\nu}{t^\nu} \displaystyle\int_0^t (t-s)^{\nu-1} g(s)ds, & \text{if } t>0. \end{cases}$$

Once we have defined the operators, we define the spaces as range spaces, as we did in Section 1.3:

Definition 2.2.1. *Let $\nu > 0$ and $1 \le p < \infty$. We define the function space $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ by*

$$\mathcal{T}_p^{(\nu)}(|t|^\nu) := \mathcal{C}^*_{\nu,\mathbb{R}}(L_p(\mathbb{R}))$$

By using similar arguments to those we used for $\mathcal{T}_p^{(\nu)}(t^\nu)$, we conclude that $\mathcal{C}^*_{\nu,\mathbb{R}}$ is an injective operator. We endow $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ with the norm

$$|||f|||_{p,(\nu)} := \|(\mathcal{C}^*_{\nu,\mathbb{R}})^{-1} f\|_p, \qquad f \in \mathcal{T}_p^{(\nu)}(|t|^\nu).$$

With this norm, $(\mathcal{C}^*_{\nu,\mathbb{R}})^{-1}$ is an isometry and therefore $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ is a Banach space.

We can state here the relation between $\mathcal{C}^*_{\nu,\mathbb{R}}$ and fractional derivatives and integrals in the whole real line. Let $L_p(\mathbb{R}, |t|^{\nu p})$ denote the Banach space of measurable functions f such that $t \mapsto |t|^\nu f(t)$ belongs to $L_p(\mathbb{R})$, and let τ_ν denote now the multiplication operator by the (weight) function $t \mapsto |t|^\nu$, $t \in \mathbb{R}$. Put $\mu_{-\nu} := \Gamma(\nu+1)\tau_{-\nu}$. Set $W^\nu_{\mathbb{R}} \colon \mathcal{T}_p^{(\nu)}(|t|^\nu) \xrightarrow{(\mathcal{C}^*_{\nu,\mathbb{R}})^{-1}} L_p(\mathbb{R}) \xrightarrow{\mu_{-\nu}} L_p(\mathbb{R}, |t|^{\nu p})$; that is,

$$W^\nu_{\mathbb{R}} f(t) := \Gamma(\nu+1)|t|^{-\nu} \left[(\mathcal{C}^*_{\nu,\mathbb{R}})^{-1} f\right](t), \quad f \in \mathcal{T}_p^{(\nu)}(|t|^\nu),\ t \in \mathbb{R}.$$

Just to clarify, we present this relation as a commutative diagram:

$$\begin{tikzcd}
\mathcal{T}_p^{(\nu)}(|t|^\nu) \arrow[r, "(\mathcal{C}^*_{\nu,\mathbb{R}})^{-1}"] \arrow[d, "W^\nu_{\mathbb{R}}"'] & L_p(\mathbb{R}) \arrow[d, "\mu_{-\nu}"] \\
L_p(\mathbb{R}, |t|^{\nu p}) \arrow[r, shift left, "\equiv"] & L_p(\mathbb{R}, |t|^{\nu p}) \arrow[l, shift left]
\end{tikzcd}$$

With inverse $W^{-\nu}_{\mathbb{R}}$:

$$\begin{tikzcd}
\mathcal{T}_p^{(\nu)}(|t|^\nu) & L_p(\mathbb{R}) \arrow[l, "\mathcal{C}^*_{\nu,\mathbb{R}}"'] \\
L_p(\mathbb{R}, |t|^{\nu p}) \arrow[u, "W^{-\nu}"] \arrow[r, shift left, "\equiv"] & L_p(\mathbb{R}, |t|^{\nu p}) \arrow[l, shift left] \arrow[u, "\Gamma(\nu+1)^{-1}\tau_\nu"']
\end{tikzcd}$$

These operators $W^\nu_{\mathbb{R}}, W^{-\nu}_{\mathbb{R}}$ are the real line versions of the Weyl fractional integrals and derivatives. In fact, since $L_p(\mathbb{R})$ is the (topological) direct sum $L_p(\mathbb{R}) = L_p(\mathbb{R}^-) \oplus L_p(\mathbb{R}^+)$ where each $f \in L_p(\mathbb{R})$ is $f = f_- + f_+$, we can write operators $\mathcal{C}^*_{\nu,\mathbb{R}}$, $\mathcal{C}_{\nu,\mathbb{R}}$ as direct sums

$$\mathcal{C}^*_{\nu,\mathbb{R}} = \mathcal{C}^*_{\nu,-} \oplus \mathcal{C}^*_{\nu,+} \qquad \text{and} \qquad \mathcal{C}_{\nu,\mathbb{R}} = \mathcal{C}_{\nu,-} \oplus \mathcal{C}_{\nu,+}$$

where each $\mathcal{C}^*_{\nu,j}$ or $\mathcal{C}^*_{\nu,j}$ acts on $L_p(\mathbb{R}^j)$ for $j \in \{-,+\}$.

Then the space $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ can be given as

$$\mathcal{T}_p^{(\nu)}(|t|^\nu) = \mathcal{T}_p^{(\nu)}((-t)^\nu) \oplus \mathcal{T}_p^{(\nu)}(t^\nu),$$

where $\mathcal{T}_p^{(\nu)}((-t)^\nu) := \mathcal{C}^*_{\nu,-}(L_p(\mathbb{R}^-))$ and $\mathcal{T}_p^{(\nu)}(t^\nu)$ is as formerly defined. ($\mathcal{C}^*_{\nu,-}$ is the equivalent Cesàro-Hardy operator for functions in $L_p(\mathbb{R}^-)$.)

We give the expression for the "negative part" of Weyl fractional integral and derivation operators since we will use them in the proof of Theorem 2.2.3. For $f \in \mathcal{T}_p^{(\nu)}(|t|^\nu)$ and $t \in \mathbb{R}$,

$$W_-^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{t} (t-s)^{\nu-1} f(s)ds,$$

$$W_-^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_{-\infty}^{x} (t-s)^{n-\nu-1} f(s)ds,$$

for every natural number $n > \nu$. $W_-^0 f = f$. Putting $\tilde{f}(t) = f(-t)$, it is readily seen that $W^\nu f(t) = W_-^\nu \tilde{f}(-t)$ for all $\nu \in \mathbb{R}$ and $t \in \mathbb{R}$. Equalities $W_-^{\nu+\mu} = W_-^\nu W_-^\mu$ on $\mathcal{S}_+$ and $W_-^n f = f^{(n)}$, $f \in \mathcal{S}_+$, hold for each natural number n and $\nu, \mu \in \mathbb{R}$.

As we did in Section 1.3 we have that $W_{\mathbb{R}}^\nu (h_\lambda) = \lambda^\nu (W_{\mathbb{R}}^\nu h)_\lambda$, where $h_\lambda(t) = h(\lambda t)$ for $t \in \mathbb{R}$ and $\lambda > 0$.

Similar properties to those of $\mathcal{T}_p^{(\nu)}(t^\nu)$ hold for $\mathcal{T}_p^{(\nu)}(|t|^\nu)$. The proof of the next proposition is similar to the proof of Proposition 1.3.2 and we skip it.

Proposition 2.2.2. *Take $p \geq 1$ and $\mu > \nu > 0$. Then*

(1) $\mathcal{T}_p^{(\mu)}(|t|^\mu) \hookrightarrow \mathcal{T}_p^{(\nu)}(|t|^\nu) \hookrightarrow L_p(\mathbb{R})$.

(2) *If $p > 1$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then the dual of $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ is $\mathcal{T}_q^{(\nu)}(|t|^\nu)$, where the duality is given by*

$$\langle f, g \rangle_\nu = \frac{1}{\Gamma(\nu+1)^2} \int_{-\infty}^{\infty} W_{\mathbb{R}}^\nu f(t) W_{\mathbb{R}}^\nu g(t) |t|^{2\nu} dt = \int_{-\infty}^{\infty} (C_{\nu,\mathbb{R}}^*)^{-1} f(t) (C_{\nu,\mathbb{R}}^*)^{-1} g(t) dt.$$

for $f \in \mathcal{T}_p^{(\nu)}(|t|^\nu)$, $g \in \mathcal{T}_q^{(\nu)}(|t|^\nu)$.

For $p = 1$, the subspace $\mathcal{T}_1^{(\nu)}(|t|^\nu)$ was introduced in [GM, Definition 1.9]. In fact $\mathcal{T}_1^{(\nu)}(|t|^\nu)$ is a subalgebra of $L_1(\mathbb{R})$ for the convolution product

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s) g(s) ds, \qquad t \in \mathbb{R}, \quad f, g \in \mathcal{T}_1^{(\nu)}(|t|^\nu), \tag{2.8}$$

see [GM, Theorem 1.8] and also [Mi2, Theorem 2] for some more details.

Next we prove that $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ is a module for the Banach algebra $\mathcal{T}_1^{(\nu)}(|t|^\nu)$. It is the analogue of Proposition 1.3.2 (2), which will be generalized in Chapter 5 (Theorem 5.1.15). We include the proof here to ilustrate that working on the whole real line in this case can be reduced to the half-line.

Theorem 2.2.3. *Let $1 \leq p < \infty$. The Banach space $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ is a module for the algebra $\mathcal{T}_1^{(\nu)}(|t|^\nu)$ and*

$$|||f * g|||_{p,(\nu)} \leq C_{p,\nu} |||f|||_{p,(\nu)} |||g|||_{1,(\nu)}, \qquad f \in \mathcal{T}_p^{(\nu)}(|t|^\nu), \quad g \in \mathcal{T}_1^{(\nu)}(|t|^\nu).$$

Proof. Take $f, g \in \mathcal{S}$. We write $f_+ := f\chi_{[0,\infty)}$ and $f_- := f\chi_{(-\infty,0]}$. By considering the decomposition $f * g = (f_+ * g_+) + (f_+ * g_-) + (f_- * g_+) + (f_- * g_-)$ on $\mathbb{R}$, we apply [GM, Lemma 1.6] and the fact that $f_- * g_- = 0$ on $(0,\infty)$ to obtain that

$$W^\nu(f*g)_+(t) = W^\nu(f_+ * g_+)(t) + (W^\nu f_+ * g_-)(t) + (W_+^\nu g_+ * f_-)(t), \qquad t > 0.$$

Now, first,

$$\|f_+ * g_+\|_{p,(\nu)} \le C_{p,\nu}\|f_+\|_{p,(\nu)}\|g_+\|_{1,(\nu)} \le C_{p,\nu}|||f|||_{p,(\nu)}|||g|||_{1,(\nu)}$$

by Proposition 1.3.2 (2).

On the other hand, $\mathcal{T}_1^{(\nu)}(t^\nu) \subset L_1(\mathbb{R}^+)$, and we apply the Minkowski inequality to get

$$\begin{aligned}
\left(\int_0^\infty |W^\nu f_+ * g_-(t)|^p t^{\nu p} dt\right)^{\frac{1}{p}} &\le \left(\int_0^\infty \left(\int_0^\infty |W^\nu f_+(s+t)||g_-(s)|ds\right)^p t^{\nu p} dt\right)^{\frac{1}{p}} \\
&= \int_0^\infty |g_-(s)| \left(\int_0^\infty |W^\nu f_+(t+s)|^p t^{\nu p} dt\right)^{\frac{1}{p}} ds \\
&\le \int_0^\infty |g_-(s)| \left(\int_s^\infty |W^\nu f_+(u)|^p u^{\nu p} du\right)^{\frac{1}{p}} ds \\
&\le \Gamma(\nu+1)|||g|||_{1,(0)}||f_+||_{p,(\nu)} \le \Gamma(\nu+1)|||g|||_{1,(\nu)}|||f|||_{p,(\nu)}.
\end{aligned}$$

As $\mathcal{T}_p^{(\nu)}(t^\nu) \subset L_p(\mathbb{R}^+)$ for $p > 1$, applying again the Minkowski inequality, we obtain

$$\begin{aligned}
\left(\int_0^\infty |(W^\nu g_+ * f_-)(t)|^p t^{\nu p} dt\right)^{\frac{1}{p}} &\le \left(\int_0^\infty \left(\int_t^\infty |W^\nu g_+(s)||f_-(t-s)|ds\right)^p t^{\nu p} dt\right)^{\frac{1}{p}} \\
&= \int_0^\infty |W^\nu g_+(s)| \left(\int_0^s |f_-(t-s)|^p t^{\nu p} dt\right)^{\frac{1}{p}} ds \\
&\le |||f|||_{p,(0)} \int_0^\infty |W^\nu g_+(s)| s^\nu ds \\
&\le \Gamma(\nu+1)|||f|||_{p,(\nu)}\,||g_+||_{1,(\nu)} \le \Gamma(\nu+1)|||f|||_{p,(\nu)}\,|||g|||_{1,(\nu)}.
\end{aligned}$$

By combination of the estimates obtained above, one gets

$$\frac{1}{\Gamma(\nu+1)}\left(\int_0^\infty |W^\nu(f*g)(t)|^p\, t^{\nu p} dt\right)^{\frac{1}{p}} \le C|||f|||_{p,(\nu)}\,|||g|||_{1,(\nu)}.$$

Finally, since $W_-^\nu(f*g)(t) = W^\nu(\tilde{f} * \tilde{g})(-t)$ if $t < 0$ and $\mathcal{T}_p^{(\nu)}(t^\nu) \subset L_p(\mathbb{R}^+)$, we have

$$\frac{1}{\Gamma(\nu+1)}\left(\int_{-\infty}^0 |W_-^\nu(f*g)(t)|^p\, |t|^{\nu p}\, dt\right)^{\frac{1}{p}} \le C|||f|||_{p,(\nu)}\,|||g|||_{1,(\nu)}.$$

The result follows. $\square$

2.2.2 Composition group on Sobolev spaces defined on $\mathbb{R}$

We remark that, as in the case of $\mathcal{T}_p^{(\nu)}(t^\nu)$, it is easy to verify that $(T_p(t))_{t\in\mathbb{R}}$ is a C_0-group of isometries on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ as the next theorem shows. The proof runs parallel to the proofs of Theorem 2.1.2, Proposition 2.1.3 and Proposition 2.1.4 and hence we omit it.

Theorem 2.2.4. *Let $p \geq 1$ and $\nu \geq 0$. We define the family of operators $(T_p(t))_{t\in\mathbb{R}}$ by*

$$T_p(t)f(s) := e^{-t/p}f(e^{-t}s), \qquad f \in \mathcal{T}_p^{(\nu)}(|t|^\nu).$$

(1) *Then $(T_p(t))_{t\in\mathbb{R}}$ is a C_0-group of isometries on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ whose infinitesimal generator Λ_p is given by*

$$(\Lambda_p f)(s) := -sf'(s) - \frac{1}{p}f(s)$$

with domain $D(\Lambda_p) = \mathcal{T}_p^{(\nu+1)}(|t|^{\nu+1})$.

(2) *$\sigma_\pi(\Lambda_p) = \varnothing$ and $\sigma(\Lambda_p) = i\mathbb{R}$.*

(3) *The semigroups $(T_p(t))_{t\geq 0}$ and $(T_q(-t))_{t\geq 0}$ are dual operators of each other acting on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ and $\mathcal{T}_q^{(\nu)}(|t|^\nu)$ with $\frac{1}{p}+\frac{1}{q} = 1$ for $p > 1$.*

2.2.3 Generalized Cesàro operators on Sobolev spaces defined on $\mathbb{R}$

In subsection 2.2.1 we have defined spaces $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ as range of operators $\mathcal{C}^*_{\nu,\mathbb{R}}$. Now we consider how does these operators behave acting on the spaces. As we did in subsection 2.1.2, we use subindex β for the operators and ν for the spaces, emphasizing that there is no required correlation between them.

We collect many results which are the analogues to those presented in subsection 2.1.2. We give them without a proof.

Recall that in subsection 2.1.1 we introduced an operator related with fractional integro-differentiation. We adapt it to our setting: let $\mathcal{W}^\nu_{\mathbb{R}} = \tau_\nu \circ W^\nu_{\mathbb{R}}$, that is, $\mathcal{W}^\nu_{\mathbb{R}}f(t) := |t|^\nu W^\nu_{\mathbb{R}}f(t)$. Note that $\mathcal{W}^\nu : \mathcal{T}_p^{(\nu)}(|t|^\nu) \to L_p(\mathbb{R})$.

Theorem 2.2.5. *Let $\nu \geq 0$, $\beta > 0$ and $1 < p < \infty$. Then*

(1) *The operator $\mathcal{C}_{\beta,\mathbb{R}}$ is bounded on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ and*

$$||\mathcal{C}_\beta|| = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}.$$

(2)

$$\sigma(\mathcal{C}_\beta) = \Gamma(\beta+1)\overline{\left\{\frac{\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)}\ :\ t\in\mathbb{R}\right\}}.$$

Theorem 2.2.6. *Let $\nu \geq 0$, $\beta > 0$ and $1 \leq p < \infty$. Then*

(1) *The operator $\mathcal{C}^*_{\beta,\mathbb{R}}$ is bounded on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ and*

$$||\mathcal{C}^*_{\beta,\mathbb{R}}|| = \frac{\Gamma(\beta+1)\Gamma(1/p)}{\Gamma(\beta+1/p)}.$$

(2) *The dual operator of $\mathcal{C}_{\beta,\mathbb{R}}$ on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$ is $\mathcal{C}^*_{\beta,\mathbb{R}}$ on $\mathcal{T}_q^{(\nu)}(|t|^\nu)$, i.e.*

$$\langle \mathcal{C}_{\beta,\mathbb{R}} f, g\rangle_\nu = \langle f, \mathcal{C}^*_{\beta,\mathbb{R}} g\rangle_\nu, \qquad f \in \mathcal{T}_p^{(\nu)}(|t|^\nu), \quad g \in \mathcal{T}_q^{(\nu)}(|t|^\nu),$$

where q and p are conjugate exponents and $\langle\ ,\ \rangle_\nu$ is given in Proposition 2.2.2 (2).

(3)

$$\sigma(\mathcal{C}^*_\beta) = \Gamma(\beta+1)\overline{\left\{\frac{\Gamma(\frac{1}{p}+it)}{\Gamma(\beta+\frac{1}{p}+it)}\ :\ t \in \mathbb{R}\right\}}.$$

2.3 Fourier transform and Cesàro-Hardy operators

In the next theorem, we consider the Fourier transform $\mathcal{F}$ on the Sobolev space $\mathcal{T}_p^{(n)}(|t|^n)$.

Theorem 2.3.1. *Take $1 \leq p \leq 2$ and $n \in \mathbb{N}$. Then $\mathcal{F}f \in \mathcal{T}_q^{(n)}(|t|^n)$ for $f \in \mathcal{T}_p^{(n)}(|t|^n)$ and $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. Take $f \in \mathcal{T}_p^{(n)}(|t|^n)$. Since $\mathcal{T}_p^{(n)}(|t|^n) \subset \mathcal{T}_p^{(j)}(|t|^j)$, we have that $x^j f^{(j)} \in L_p(\mathbb{R})$ for $0 \leq j \leq n$. As

$$(it)^n(\mathcal{F}f)^{(n)}(t) = \sum_{j=0}^n (-1)^n \binom{n}{j}\frac{n!}{j!}\mathcal{F}(x^j f^{(j)})(t), \quad n \in \mathbb{N},\ t \text{ a.e. on } \mathbb{R},$$

(see for example [Z]), we conclude that $(it)^n(\mathcal{F}f)^{(n)} \in L_q(\mathbb{R})$ and then $\mathcal{F}f \in \mathcal{T}_q^{(n)}(|t|^n)$. □

Analogously to what has been done with $\mathcal{L}$ in Lemma 1.1.3 and Corollary 1.2.6, next we will see that

$$\mathcal{F}(\mathcal{C}_{\beta,\mathbb{R}} f) = \mathcal{C}^*_{\beta,\mathbb{R}}(\mathcal{F}f), \qquad \text{and} \qquad \mathcal{F}(\mathcal{C}^*_{\beta,\mathbb{R}} f) = \mathcal{C}_\beta(\mathcal{F}f), \qquad f \in L_p(\mathbb{R}),$$

for $1 < p \leq 2$ (Theorem 2.3.4). This theorem extends the case $\beta = 1$ formulated in [Be] and proved in [Mo2]. Our approach is based in the subordination of $\mathcal{C}_{\beta,\mathbb{R}}$ and $\mathcal{C}^*_{\beta,\mathbb{R}}$ to the group $(T_p(t))_{t\in\mathbb{R}}$. In Theorem 2.2.4 (1) we have stated that it is a C_0-group of isometries on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$, it is straightforward that it is also a C_0-group on $L_p(\mathbb{R})$.

Lemma 2.3.2. *Let $1 \le p \le 2$ and q its conjugate exponent. Then*

$$\mathcal{F} \circ T_p(t) = T_q(-t) \circ \mathcal{F}, \qquad \text{on } L_p(\mathbb{R}).$$

Proof. Consider $1 \le p \le 2$ and $f \in \mathcal{S}$. It is clear that $T_p(t)f \in \mathcal{S}$. Note that

$$\begin{aligned} \mathcal{F}(T_p(t)f)(r) &= e^{-t/p}\int_{-\infty}^{\infty} e^{-irx} f(e^{-t}x)dx = e^{t(1-\frac{1}{p})}\int_{-\infty}^{\infty} e^{-ire^t y} f(y)dy \\ &= e^{\frac{t}{q}}\mathcal{F}f(e^t r) = T_q(-t)(\mathcal{F}f)(r). \end{aligned}$$

By density of $\mathcal{S}$ we conclude the result. $\square$

Remark 2.3.3. Since $\mathcal{T}_p^{(\nu)}(|t|^\nu) \hookrightarrow L_p(\mathbb{R})$ (Proposition 2.2.2 (1)), the equality $\mathcal{F}(T_p(t)f) = T_q(-t)(\mathcal{F}f)$ holds for $f \in \mathcal{T}_p^{(\nu)}(|t|^\nu)$ for $\nu \ge 0$ and $1 \le p \le 2$.

Finally, we are ready to prove the main result in this section.

Theorem 2.3.4. *Let $\beta > 0$.*

(1) *If $f \in L_p(\mathbb{R})$ for some $1 < p \le 2$, then $\mathcal{F}(\mathcal{C}_{\beta,\mathbb{R}}f) = \mathcal{C}^*_{\beta,\mathbb{R}}(\mathcal{F}f)$.*

(2) *If $f \in L_p(\mathbb{R})$ for some $1 \le p \le 2$, then $\mathcal{F}(\mathcal{C}^*_{\beta,\mathbb{R}}f) = \mathcal{C}_{\beta,\mathbb{R}}(\mathcal{F}f)$.*

Proof. (1) Take $f \in L_p(\mathbb{R})$ for some $1 < p \le 2$. By (2.7) and Lemma 2.3.2 we have that

$$\begin{aligned} \mathcal{F}(\mathcal{C}_{\beta,\mathbb{R}}f) &= \mathcal{F}\left(\int_0^\infty \varphi_{\beta,p}(r)T_q(r)f dr\right) = \int_0^\infty \varphi_{\beta,p}(r)\mathcal{F}(T_q(r)f)dr \\ &= \int_0^\infty \varphi_{\beta,p}(r)T_p(-r)(\mathcal{F}f)dr = \mathcal{C}^*_{\beta,\mathbb{R}}(\mathcal{F}f). \end{aligned}$$

(2) Now take $f \in L_p(\mathbb{R})$ for some $1 \le p \le 2$. By (2.6) and Lemma 2.3.2 we have that

$$\begin{aligned} \mathcal{F}(\mathcal{C}^*_{\beta,\mathbb{R}}f) &= \mathcal{F}\left(\int_0^\infty \varphi_{\beta,p}(r)T_p(-r)f dr\right) = \int_0^\infty \varphi_{\beta,p}(r)\mathcal{F}(T_p(-r)f)dr \\ &= \int_0^\infty \varphi_{\beta,p}(r)T_q(r)(\mathcal{F}f)dr = \mathcal{C}_{\beta,\mathbb{R}}(\mathcal{F}f). \end{aligned}$$

$\square$

Remark 2.3.5. By (1.13), we get $\mathcal{F} \circ \mathcal{C}_{\beta,\mathbb{R}} = \mathcal{C}^*_{\beta,\mathbb{R}} \circ \mathcal{F}$ and $\mathcal{F} \circ \mathcal{C}^*_{\beta,\mathbb{R}} = \mathcal{C}_{\beta,\mathbb{R}} \circ \mathcal{F}$ on $\mathcal{T}_p^{(\nu)}(|t|^\nu)$, $1 < p \le 2$ and $\nu \ge 1$.

Chapter 3
Sobolev type algebras

In Chapter 1 we have introduced the spaces $\mathcal{T}_p^{(\nu)}(t^\nu)$ and stated that $\mathcal{T}^{(\nu)}(t^\nu)(:= \mathcal{T}_1^{(\nu)}(t^\nu))$ is a Banach algebra. In fact, the space $\mathcal{T}^{(\nu)}(t^\nu)$ has been studied as a semisimple Banach algebra in a series of papers including [GMR2], [GW2], [GW1], [GMR1], [GMM], [GMSt]. In most of the above references the properties and applications of these algebras, and of their associated mathematical objects, are very much like the ones of the Banach algebra $L_1(\mathbb{R}^+)$.

We now consider the more general case of spaces $\mathcal{T}^{(\nu)}(t^\nu\omega)$, where ω is a continuous increasing submultiplicative weight. For such weights, the spaces $\mathcal{T}^{(\nu)}(t^\nu\omega)$ are indeed Banach algebras [GM]. Beyond this fact, it is shown that the Gelfand theory of those algebras, in the semisimple case, is the same as that one of $L_1(\omega)$. In contrast, it is shown, in the second section, that for radical weights ω, $\mathcal{T}^{(\nu)}(t^\nu\omega)$ is not even an algebra. Nevertheless one can exhibit a radical setting (as presented in [GS]) by appealing to quotient algebras of Volterra type. We study some problems in this context.

3.1 The semisimple case

3.1.1 Spectrum and Gelfand transform of the algebra $\mathcal{T}^{(\nu)}(t^\nu\omega)$

In this monograph we are assuming as well known some basic aspects on commutative Banach algebras theory. We briefly recall that the *spectrum* $Spec(\mathcal{A})$ of a Banach algebra $\mathcal{A}$ is by definition the set of characters of that algebra, i.e., the set of every non-zero continuous multiplicative algebra homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ (equivalently, $Spec(\mathcal{A})$ can also be viewed as the set of all maximal regular ideals of $\mathcal{A}$). Therefore $Spec(\mathcal{A}) \subseteq \mathcal{A}'$. It turns out that $Spec(\mathcal{A})$ is a locally compact (Hausdorff) space with respect to the weak topology induced by the dual pair $(\mathcal{A}', \mathcal{A})$. This topology is called the Gelfand topology on $Spec(\mathcal{A})$. Recall that if $\mathcal{A}$ has an identity, then $Spec(\mathcal{A})$ is compact.

The mapping

$$\begin{array}{rccc} \mathcal{G} = \widehat{\cdot}\ : & \mathcal{A} & \longrightarrow & \mathcal{C}_0(Spec(\mathcal{A})) \\ & a & \longmapsto & \widehat{a} \end{array}$$

where $\widehat{a}(\varphi) = \varphi(a)$ for all $\varphi \in Spec(\mathcal{A})$, is a Banach algebra continuous homomorfism, called *Gelfand transform* of $\mathcal{A}$, which becomes a fundamental concept in the theory. A

Banach algebra $\mathcal{A}$ is called *semisimple* if Ker$\mathcal{G} = \{0\}$, and it is called *radical* if the set of non-zero characters of the spectrum of $\mathcal{A}$ is empty, i.e. if Ker$\mathcal{G} = Spec\mathcal{A}$

Let us consider the case $\mathcal{A} = L_1(\omega)$. Here ω is a weight function, with the meaning that $\omega : (0,\infty) \to (0,\infty)$ is measurable, not necessarily continuous that satisfies

$$\omega(s+t) \le \omega(s)\omega(t), \qquad s,t \in \mathbb{R}^+.$$

Set $\rho_\omega := \lim_{t\to\infty} \omega(t)^{\frac{1}{t}}$. When $\rho_\omega > 0$, we define $\sigma_\omega = -\log \rho_\omega$. The following theorem is well known (see [D1, p.189], [D2, p.530]).

Theorem. *Let ω be a weight function on $\mathbb{R}^+$.*

(1) *If $\rho_\omega > 0$, then $L_1(\omega)$ is a semisimple Banach algebra, and its character space can be identified with $\overline{\Pi}_{\sigma_\omega}$, in the sense that each character has the form*

$$\begin{array}{cccc} \varphi_z : & L_1(\omega) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & \varphi_z(f) = \mathcal{L}(f)(z) \end{array}$$

for some $z \in \overline{\Pi}_{\sigma_\omega}$.

(2) *If $\rho_\omega = 0$, then $L_1(\omega)$ is a radical Banach algebra.*

Therefore, according to the above theorem, the Gelfand transform of $L_1(\omega)$ for $\rho_\omega > 0$ coincides with the Laplace transform $\mathcal{L}$, that satisfies

$$\mathcal{L}(f * g) = (\mathcal{L}f)(\mathcal{L}g), \qquad f,g \in L_1(\mathbb{R}^+).$$

We are going to see that the Gelfand theory of algebras $\mathcal{T}^{(\nu)}(t^\nu\omega)$ is the analogue to the one of $L_1(\omega)$.

3.1.2 Characters of the algebra $\mathcal{T}^{(\nu)}(t^\nu\omega)$

We define $\mathcal{T}^{(\nu)}(t^\nu\omega)$ as a weighted analogue of $\mathcal{T}^{(\nu)}(t^\nu)$. We know that $\mathcal{C}_c^{(\infty)}[0,\infty)$ is dense in $\mathcal{T}_p^{(\nu)}(t^\nu)$, so an equivalent definition of $\mathcal{T}_p^{(\nu)}(t^\nu)$ is

$$\mathcal{T}_p^{(\nu)}(t^\nu) = \overline{\mathcal{C}_c^{(\infty)}[0,\infty)}^{\|\cdot\|_{p,(\nu)}}$$

with $\|\cdot\|_{p,(\nu)}$ given in (1.9) or (1.11).

In analogy, we define $\mathcal{T}_p^{(\nu)}(t^\nu\omega)$ as the completion of $\mathcal{C}_c^{(\infty)}[0,\infty)$ in the norm

$$\|f\|_{p,(\nu),\omega} := \left(\int_0^\infty |W^\nu f(t)|^p t^{\nu p}\omega^p(t)\right)^{\frac{1}{p}}.$$

(See [GM]).

Similarly to the properties we have seen in Proposition 1.3.2, we have the continuous inclusion $\mathcal{T}^{(\nu)}(t^\nu\omega) \overset{i}{\hookrightarrow} L_1(\omega)$, and so the Laplace transform

$$\mathcal{L} \ : \ \mathcal{T}^{(\nu)}(t^\nu\omega) \overset{i}{\hookrightarrow} L_1(\omega) \xrightarrow{\mathcal{L}} \mathbb{C}$$

is well defined. In this way, if $z \in \overline{\Pi}_{\sigma_\omega}$, the map

$$\begin{array}{rccc} \varphi_z : & \mathcal{T}^{(\nu)}(t^\nu\omega) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & \varphi_z(f) := \mathcal{L}f(z) \end{array}$$

is not identically zero, is bounded, multiplicative and linear, i.e, it is a character of $\mathcal{T}^{(\nu)}(t^\nu\omega)$. (We have considered only the half-plane $\overline{\Pi}_{\sigma_\omega}$, where

$$\sigma_\omega := -\lim_{t\to\infty} \frac{\log\omega(t)}{t},$$

to make the Laplace transform well defined, since for $\kappa > \sigma_\omega$ we have $e^{-\kappa t} < \omega(t)$). Moreover, if $z, z' \in \overline{\Pi}_{\sigma_\omega}$ with $z \neq z'$ then $\varphi_z \neq \varphi_{z'}$. Now we want to see that each character of $\mathcal{T}^{(\nu)}(t^\nu\omega)$ has the form φ_z for some $z \in \overline{\Pi}_{\sigma_\omega}$. We give the proof using Riesz kernels. Let φ be a character of $\mathcal{T}^{(\nu)}(t^\nu\omega)$. Take $f \in \mathcal{T}^{(\nu)}(t^\nu\omega)$ such that $\varphi(f) \neq 0$. For each $t > 0$, define the map

$$\begin{array}{rccc} \Phi : & (0,\infty) & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \Phi(t) := \frac{\varphi(R_t^{\nu-1} * f)}{\varphi(f)}. \end{array}$$

By [GM, p. 17],

- $R_t^{\nu-1} * f \in \mathcal{T}^{(\nu)}(t^\nu\omega), \qquad$ for $\ f \in \mathcal{T}^{(\nu)}(t^\nu\omega)$.
- $\|R_t^{\nu-1} * f\|_{1,(\nu),\omega} \le C_{\nu,\omega} t^\nu \|f\|_{(\nu)}$.
- the map

$$\begin{array}{ccc} (0,\infty) & \longrightarrow & \mathcal{T}^{(\nu)}(t^\nu\omega) \\ t & \longmapsto & R_t^{\nu-1} * f \end{array}$$

  is continuous for each $f \in \mathcal{T}^{(\nu)}(t^\nu\omega)$.

Hence we conclude that the map Φ is well defined and continuous, and it does not depend on f. Besides this,

$$|\Phi(t)| \le K t^\nu \omega(t), \qquad t > 0.$$

The next lemmata appear in [R], we include the proofs here for ease of reading.

Lemma 3.1.1. *[R, p.85] Let $s, t > 0$. Then*

$$\Phi(s)\Phi(t) = \frac{1}{\Gamma(\nu)} \left(\int_t^{s+t} - \int_0^s \right) (s + r - t)^{\nu-1} \Phi(r) dr.$$

Proof. As $(R_t^{\nu-1})_{t>0}$ is an integrated semigroup, it verifies that

$$(R_t^{\nu-1} * R_s^{\nu-1}) * f = \frac{1}{\Gamma(\nu)} \left(\int_s^{t+s} - \int_0^t \right) (t+s-u)^{\nu-1} R_u^{\nu-1} * f du$$

for each $t, s > 0$ and $f \in \mathcal{C}_c^{(\infty)}[0,\infty)$ (see [ABHN], [KuS]). Then, for $\varphi(f) \neq 0$,

$$\begin{aligned} \Phi(s)\Phi(t) &= \frac{\varphi(R_t^{\nu-1} * R_s^{\nu-1} * f)}{\varphi(f)} \\ &= \frac{1}{\Gamma(\nu)} \left(\int_s^{t+s} - \int_0^t \right) (t+s-u)^{\nu-1} \frac{\varphi(R_u^{\nu-1} * f)}{\varphi(f)} du \\ &= \frac{1}{\Gamma(\nu)} \left(\int_s^{t+s} - \int_0^t \right) (t+s-u)^{\nu-1} \Phi(u) du. \end{aligned}$$

□

(Lemma 3.1.1 suggests calling Φ an *integrated character* of order ν.) Recall that Φ is continuous and it satisfies $|\Phi(t)| \leq Kt^\nu \omega(t)$, for $t > 0$. These conditions imply that the holomorphic function given by

$$R(z) := z^\nu \int_0^\infty \Phi(t) e^{-zt} dt, \qquad z \in \overline{\Pi}_{\sigma_\omega}$$

is a pseudo-resolvent, i.e., it satisfies equation

$$R(z_1) - R(z_2) = (z_2 - z_1) R(z_1) R(z_2), \qquad z_1, z_2 \in \overline{\Pi}_{\sigma_\omega}$$

([Hi2, Proposition 2.1]). We can associate a unique complex number of $\overline{\Pi}_{\sigma_\omega}$ to this function R.

Lemma 3.1.2. *[R, p.86] There exists $z_0 \in \overline{\Pi}_{\sigma_\omega}$ such that*

$$R(z) = \frac{1}{z + z_0}.$$

Proof. Let $\lambda_0 := 1 + \sigma_\omega$. For all $z \in \overline{\Pi}_{\sigma_\omega}$ we have

$$R(\lambda_0) - R(z) = (z - \lambda_0) R(\lambda_0) R(z),$$

therefore

$$R(z) = \frac{1}{z + \left(\frac{1}{R(\lambda_0)} - \lambda_0 \right)}.$$

Take

$$z_0 := \frac{1}{R(\lambda_0)} - \lambda_0,$$

and we have the result. □

Lemma 3.1.3. *[R, p.87] Let Φ, z_0 be as in the preceding lemmata. Then*

$$\Phi(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} e^{-z_0 s} ds, \qquad t > 0.$$

Proof. Let $t > 0$ and $z_0 \in \mathbb{C}^+$. By using Fubini's theorem and a change of variables $(r - s = u)$ we get

$$\begin{aligned}\int_0^\infty \left(\frac{1}{\Gamma(\nu)} \int_0^r (r-s)^{\nu-1} e^{-z_0 s} ds \right) e^{tr} dr &= \int_0^\infty \frac{1}{\Gamma(\nu)} \int_0^\infty u^{\nu-1} e^{-tu} du e^{-(z_0+t)s} ds \\ = \frac{1}{t^\nu} \frac{1}{z_0 + t} \quad = \quad \frac{1}{t^\nu} R(t) &= \int_0^\infty \Phi(r) e^{-tr} dr \quad = \quad \mathcal{L}(\Phi)(t),\end{aligned}$$

and by injectivity of the Laplace transform we end the proof. □

Let $\mathcal{A}_0(\Pi_{\sigma_\omega})$ denote the space

$$\mathcal{A}_0(\Pi_{\sigma_\omega}) := \left\{ F \in \mathcal{C}(\overline{\Pi}_{\sigma_\omega}) \cap \mathcal{H}ol(\Pi_{\sigma_\omega}) \ : \ \lim_{\substack{\mathrm{Re}\, z \geq \sigma_\omega \\ z \to \infty}} F(z) = 0 \right\}.$$

It is well known that $\mathcal{A}_0(\Pi_{\sigma_\omega})$ is a Banach algebra for pointwise product and supremum norm on $\overline{\Pi}_{\sigma_\omega}$, and that $\mathcal{L}(L_1(\omega))$ is (densely) contained in $\mathcal{A}_0(\Pi_{\sigma_\omega})$.

Theorem 3.1.4. *The character set of the Banach algebra $\mathcal{T}^{(\nu)}(t^\nu \omega)$ has the form*

$$\{\varphi_z \ : \ z \in \overline{\Pi}_{\sigma_\omega}\}$$

in such a way that the Gelfand transform on $\mathcal{T}^{(\nu)}(t^\nu\omega)$ is given by the Laplace transform

$$\begin{aligned}\mathcal{L}: \ \mathcal{T}^{(\nu)}(t^\nu \omega) \ &\longrightarrow \ \mathcal{A}_0(\Pi_{\sigma_\omega}) \\ f \ &\longmapsto \ \mathcal{L}(f)\end{aligned}$$

Proof. We have seen before that each φ_z defines a character of $\mathcal{T}^{(\nu)}(t^\nu\omega)$. Now, let φ be a fixed character of $\mathcal{T}^{(\nu)}(t^\nu\omega)$. Let $f, g \in \mathcal{T}^{(\nu)}(t^\nu\omega)$, with $\varphi(g) \neq 0$. We have

$$\begin{aligned}\varphi(f) &= \frac{\varphi(f*g)}{\varphi(g)} = \frac{1}{\varphi(g)} \varphi \left(\int_0^\infty W^\nu f(t) R_t^{\nu-1} * g dt \right) = \int_0^\infty W_f^\nu(t) \frac{\varphi(R_t^{\nu-1} * g)}{\varphi(g)} dt \\ &= \int_0^\infty W^\nu f(t) \Phi(t) dt \ = \ \int_0^\infty \left(\frac{1}{\Gamma(\nu)} \int_s^\infty (t-s)^{\nu-1} W^\nu f(t) dt \right) e^{-z_0 s} ds \\ &= \int_0^\infty f(s) e^{-z_0 s} ds \ = \ \mathcal{L}(f)(z_0) \ = \ \varphi_{z_0}(f),\end{aligned}$$

as we wanted to show. □

Remark 3.1.5. At this point, we have proved that the algebra $\mathcal{T}^{(\nu)}(t^\nu\omega)$ is semismiple, because $\mathcal{L}$ is injective and then

$$\mathrm{Ker}\mathcal{G} \ = \ \mathrm{Ker}\mathcal{L} \ = \ \{0\}.$$

3.2 The radical case: a Sobolev algebra of Volterra type

We have just seen that, if ω satisfies $\rho_\omega := \lim_{t\to\infty} \omega(t)^{1/t} \neq 0$, then the convolution Banach algebra $\mathcal{T}^{(\nu)}(t^\nu\omega)$, $\nu > 0$, is *semisimple* and its Gelfand transform is equal to the Laplace transform (on the half-plane $\operatorname{Re} z \geq -\log \rho_\omega$). This fact was well known in the case $\nu = 0$, that is, for $L_1(\omega)$ [D2, Theorem 4.7.27 (i)].

On the other hand, it is also well known that, provided ω is *radical*, i. e., $\rho_\omega = 0$, then the Banach algebra $L_1(\omega)$ is radical, which is to say that $L_1(\omega)$ has no non-zero character or, equivalently, that the set of modular maximal ideals of $L_1(\omega)$ is empty; see [D2, Theorem 4.7.27 (ii)]. A standard and important example of radical weight is $\omega(t) := e^{-t^2}$, $t > 0$.

Radical Banach algebras are known from the very beginning of the theory of Banach algebras, but they were not studied in depth until quite recently. The modern interest in such algebras emerges with the solution to the Kaplansky problem obtained independently by H. G. Dales and J. Esterle. They proved that, given an infinite compact space K, and assuming the continuum hypothesis, there always exists a discontinuous injective homomorphism $\theta\colon C(K) \to R \oplus \mathbb{C}$, for suitable commutative radical Banach algebras R. Here, $C(K)$ is the usual Banach algebra of complex continuous functions on K, and one can take as R the weighted algebra $L^1(e^{-t^2})$ or the Volterra algebra $L^1_*(0,1)$; see [DE] for a joint presentation of the Dales-Esterle theorem. This result has been subsequently extended or complemented in several directions. For instance (always under the continuum hypothesis), algebras $L_1(\omega)$, for ω radical, are universal in the class of complex and commutative algebras with no unit which are integral domains and have power of the continuum; see [E1, Cor. 5.2]. In another direction, Esterle characterizes *all* the radical Banach algebras R for which it is possible to construct a discontinuous homomorphism $\theta\colon C(K) \to R \oplus \mathbb{C}$ (under the assumption of the continuum hypothesis again); see [E2, Th. 6.4] and also [E3, Th. 5.3]. Further, these algebras form the class number 5 of a total of 9 which have been introduced in [E3] as a way to classify the set of commutative radical Banach algebras. Most of the (rich set of) examples and counterexamples given in [E3] are constructed out from convolution algebras of ℓ^1 or L^1 type, or principal ideals of them.

In accordance with all the above considerations, it sounds sensible to investigate convolution radical Banach algebras of Sobolev type in the above setting, that is, associated and in analogy with radical algebras like $L_1(\omega)$ or $L^1_*(0,1)$. Thus our first question is to know if, similarly to the semisimple case, the Banach algebra $\mathcal{T}^{(\nu)}(t^\nu\omega)$, for $\nu > 0$, is radical whenever ω is a radical weight.

In a somehow disappointing way, it turns out that $\mathcal{T}^{(\nu)}(t^\nu\omega)$ need not be a convolution algebra if one allows ω to be a decreasing function (see Section 3.2.1). Thus, in order to find radical Banach algebras of Sobolev type, one must try some other way different from the one suggested by the (continuous) inclusion $\mathcal{T}^{(\nu)}(t^\nu\omega) \hookrightarrow L_1(\omega)$.

Recall that the convolution Volterra algebra $L^1_*(0,1)$ formed by all Borel measurable functions $f\colon (0,1) \to \mathbb{C}$ such that $\|f\|_1 = \int_0^1 |f(t)|dt < \infty$, endowed with the convolution product, can be represented as the quotient $L^1_*(0,1) \cong L_1(\mathbb{R}^+)/\mathcal{I}_1$, where $\mathcal{I}_1$ is the closed ideal

$$\mathcal{I}_1 = \left\{ f \in L_1(\mathbb{R}^+) \ : \ f \equiv 0 \text{ a.e. on } (0,1) \right\}.$$

Similarly, let us consider the quotient $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_1^{(\nu)}$, for the closed ideal $\mathcal{I}_1^{(\nu)} := \mathcal{T}^{(\nu)}(t^\nu) \cap \mathcal{I}_1$. We show in Section 3.2.1 that it is a radical Banach algebra which is indeed topologically generated by its nilpotent elements. Since there is the identification

$$f + \mathcal{I}_1^{(\nu)} \ \longleftrightarrow \ f|_{(0,1)}, \ \mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_1^{(\nu)} \hookrightarrow L^1_*(0,1)$$

a natural question to pose in this respect is to find out which elements of $L^1_*(0,1)$ correspond to the classes $f + \mathcal{I}_1^{(\nu)}$ $(f \in \mathcal{T}^{(\nu)}(t^\nu))$. A complete answer to that problem is given in Section 3.2.2, for *integer* $\nu = n$. Namely, the quotient algebra $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_1^{(n)}$ coincides with the space $\mathcal{V}^{(n)}(0,1)$ formed by all functions $f : (0,1] \longrightarrow \mathbb{C}$ for which there exist $f, f', \cdots, f^{(n-1)}$ on $(0,1]$, $f^{(n-1)}$ is absolutely continuous on $(0,1]$, and $\int_0^1 |f^{(n)}(x)| x^n dx < \infty$. Moreover, the quotient norm in $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_1^{(\nu)}$ is equivalent to the norm

$$\|f\|_{\mathcal{V}^{(n)}(0,1)} := \int_0^1 |f^{(n)}(x)| x^n \, dx \ + \max_{0 \le i \le n-1} |f^{(i)}(1)| \ , \quad f \in \mathcal{V}^{(n)}(0,1).$$

A consequence of the above equivalence is that the space $\mathcal{V}^{(n)}(0,1)$ is a radical Banach algebra for the convolution and the above norm (Corollary 3.2.7). Thus $\mathcal{V}^{(n)}(0,1)$ is a generalization of the Volterra algebra formed by (higher order) absolutely continuous functions on $(0,1)$. The representation of the elements $f + \mathcal{I}_1^{(\nu)}$ for $f \in \mathcal{T}^{(\nu)}(t^\nu)$ and general fractional ν is quite more difficult than in the integer case $\nu \in \mathbb{N}$, and in fact it remains unsolved. We briefly discuss the reason for that at the end of Section 3.2.2.

In Section 3.2.3, on the basis of results obtained in [GW2], we show that all closed ideals of $\mathcal{V}^{(n)}(0,1)$ are standard, and then it follows from the main theorem of [JS] that all epimorphisms onto $\mathcal{V}^{(n)}(0,1)$ and all derivations from $\mathcal{V}^{(n)}(0,1)$ into itself are bounded. However, a complete characterization of the set of all such derivations has not been obtained yet. The chapter ends with a result, Corollary 3.2.15, where a fairly large class of concrete derivations of $\mathcal{V}^{(n)}(0,1)$ is given.

3.2.1 Quotient radical Sobolev algebras

As pointed out in the previous section, if ω is nondecreasing then the space $\mathcal{T}^{(\nu)}(t^\nu\omega)$ is in fact a Banach algebra for the usual convolution on $(0,\infty)$, and a subalgebra of $L_1(\omega)$; moreover it is semisimple if $\lim_{t\to\infty} \omega(t)^{1/t} \neq 0$. More details and properties of the Weyl fractional derivative and algebras $\mathcal{T}^{(\nu)}(t^\nu\omega)$ can be seen in [SKM], [GM].

Here, and in analogy to the $L_1(\omega)$ case, we would like to have that $\mathcal{T}^{(\nu)}(t^\nu\omega)$ is a radical Banach algebra when $\lim_{t\to\infty} \omega(t)^{1/t} = 0$. Unfortunately, it happens that spaces $\mathcal{T}^{(\nu)}(t^\nu\omega)$ are not in general convolution algebras for decreasing weights ω. To see this, take any weight ω such that

$$0 < c_\omega := \int_0^\infty t \ \omega(t) \, dt < \infty. \tag{3.1}$$

For $\lambda > 0$, the function $e_\lambda(t) := e^{-\lambda t}$, $(t > 0)$, belongs to $\mathcal{T}^{(1)}(t\omega)$; indeed,

$$\|e_\lambda\|_{\mathcal{T}^{(1)}(t\omega)} = \lambda \int_0^\infty te^{-\lambda t}\omega(t)\,dt \le \lambda \int_0^\infty t\,\omega(t)\,dt = \lambda c_\omega.$$

Also, a simple calculation gives us that $(e_\lambda * e_\lambda)(t) = te^{-\lambda t}$ for all $t > 0$. Hence,

$$\|e_\lambda * e_\lambda\|_{\mathcal{T}^{(1)}(t\omega)} = \int_0^\infty |1 - \lambda t|\ te^{-\lambda t}\omega(t)\,dt.$$

If $\mathcal{T}^{(1)}(t\omega)$ were a Banach algebra for the convolution, we would have, for some constant $C > 0$,

$$\|e_\lambda * e_\lambda\|_{\mathcal{T}^{(1)}(t\omega)} \le C\|e_\lambda\|^2_{\mathcal{T}^{(1)}(t\omega)};$$

that is to say,

$$\int_0^\infty |1 - \lambda t| te^{-\lambda t}\omega(t)\,dt \le C\lambda^2 c_\omega^2 \qquad (\lambda > 0),$$

but this cannot be true, since as λ tends to 0 we get $c_\omega \le 0$, a contradiction. Then it follows that $\mathcal{T}^{(1)}(t\omega)$ is not a convolution algebra.

Note that the radical weight function $\omega(t) = e^{-t^2}$ satisfies the preceding condition (3.1). Therefore $\mathcal{T}^{(1)}(te^{-t^2})$ is not even an algebra for the convolution product. So we need to look for other candidates to get convolution radical algebras involving derivatives. Let us follow the model suggested by the Volterra algebra $L^1_*(0,1)$.

Here we work with any $a > 0$ rather than merely with $a = 1$. Thus let define the subset

$$\mathcal{I}_a^{(\nu)} := \{f \in \mathcal{T}^{(\nu)}(t^\nu)\ :\ f \equiv 0 \text{ a.e. on } (0,a)\} = \{f \in \mathcal{T}^{(\nu)}(t^\nu)\ :\ \gamma(f) \ge a\},$$

where, within the second brackets, $\gamma(f) := \inf(\mathrm{supp} f)$.

Lemma 3.2.1. *$\mathcal{I}_a^{(\nu)}$ is a closed ideal of $\mathcal{T}^{(\nu)}(t^\nu)$ for all $a > 0$.*

Proof. Put $J_a := \{f \in L_1(\mathbb{R}^+)\ :\ f \equiv 0 \text{ a.e. on } (0,a)\}$. It is well known that J_a is a closed ideal of $L_1(\mathbb{R}^+)$. Thus the result is a consequence of the continuity of the inclusion mapping $\iota\colon \mathcal{T}^{(\nu)}(t^\nu) \hookrightarrow L_1(\mathbb{R}^+)$ (see [GM, p. 16]) since $\mathcal{I}_a = \iota^{-1}(J_a)$. □

We call $\mathcal{I}_a^{(\nu)}$ *standard ideal* of $\mathcal{T}^{(\nu)}(t^\nu)$ at a.

Now, let us consider the quotient Banach algebra $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$. Since density is preserved by passing to the quotient, we get the following theorem.

Theorem 3.2.2. *The ideal of its nilpotent elements is dense in $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$. Hence, the Banach algebra $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ is radical.*

Proof. For every $f \in C_c^{(\infty)}(0,\infty)$ there exists an integer N such that $\gamma(f^{*N}) > a$. Then the result follows from Proposition 1.3.2 (1) and the commutativity of the algebras. □

Remark 3.2.3. An alternative way to show that the quotient algebra $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ is radical is to check that the hull $h(\mathcal{I}_a^{(\nu)}) := \{z \in \mathbb{C} : \operatorname{Re} z > 0,\ \mathcal{L}(g)(z) = 0\ (g \in \mathcal{I}_a^{(\nu)})\}$ of the ideal $\mathcal{I}_a^{(\nu)}$ is empty. This is accomplished by a standard argument.

Remark 3.2.4. Note that the continuous inclusions

$$\mathcal{T}^{(\mu)}(t^\mu) \hookrightarrow \mathcal{T}^{(\nu)}(t^\nu) \qquad (\mu \geq \nu \geq 0)$$

are inherited by the above quotient radical Banach algebras; i. e., for $\mu \geq \nu \geq 0$,

$$\mathcal{T}^{(\mu)}(t^\mu)/\mathcal{I}_a^{(\mu)} \hookrightarrow \mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}.$$

In fact,

$$\begin{aligned}
\|f + \mathcal{I}_a^{(\nu)}\|_{\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}} &= \inf\left\{\|f+h\|_{\mathcal{T}^{(\nu)}(t^\nu)} \ : \ h \in \mathcal{I}_a \cap \mathcal{T}^{(\nu)}(t^\nu)\right\} \\
&\leq \inf\left\{\|f+h\|_{\mathcal{T}^{(\nu)}(t^\nu)} \ : \ h \in \mathcal{I}_a \cap \mathcal{T}^{(\mu)}(t^\mu)\right\} \\
&\leq C_{\nu,\mu} \inf\left\{\|f+h\|_{\mathcal{T}^{(\mu)}(t^\mu)} \ : \ h \in \mathcal{I}_a \cap \mathcal{T}^{(\mu)}(t^\mu)\right\} \\
&= C_{\nu,\mu}\|f + \mathcal{I}_a^{(\mu)}\|_{\mathcal{T}^{(\mu)}(t^\mu)/\mathcal{I}_a^{(\mu)}}.
\end{aligned}$$

The first part of the following result is not strictly necessary in the realm of this section, but we include it here for the sake of information.

Proposition 3.2.5. *For z such that $\operatorname{Re} z > 0$, let σ^z be the element of $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ defined by the function $x \mapsto \Gamma(z)^{-1}x^{z-1}$, $(x > 0)$. Then $(\sigma^z)_{\operatorname{Re} z>0}$ is an analytic semigroup in $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ for every $\nu > 0$, such that*

(1) $\sup_{t\in(0,1)} \|\sigma^t\|_{\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}} < \infty.$

(2) $\operatorname{span}\{\sigma^{\mathrm{k}} : \mathrm{k} \in \mathbb{N}\}$ *is dense in* $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$.

Proof. The assertions are readily seen from the fact that for $\operatorname{Re} z > 0$ the function $x \mapsto \Gamma(z)^{-1}x^{z-1}e^{-x}$, $x \in (0,\infty)$, satisfies in the Banach algebra $\mathcal{T}^{(\nu)}(t^\nu)$ analogue properties to those of the statement; see [GMR1, Proposition 1.1]. □

Thus the proposition tells us in particular that the subspace of polynomials is dense in $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$, and that $(\sigma^t)_{0<t<1}$ is a bounded approximate identity (b. a. i., for short) for $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ (there are many more b. a. i. in $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ according to the proof of Proposition 1.3.2).

The mapping $f + \mathcal{I}_a^{(\nu)} \mapsto f|_{(0,a)}$, $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)} \hookrightarrow L^1_*(0,a)$ is obviously well defined. The image of that mapping is studied in the next section.

3.2.2 Representation of $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}$ on $(0,a]$

The task of representing the elements $f+\mathcal{I}_a^{(\nu)}$, for $f\in\mathcal{T}^{(\nu)}(t^\nu)$, as functions a. e. defined on the interval $(0,a)$ looks complicated for general fractional ν (see the short discussion in Remark 3.2.8 at the end of this section). Here we settle the question for integer order of derivation.

For $f\in\mathcal{T}^{(n)}(t^n)$, let $\|f+\mathcal{I}_a^{(n)}\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}}$ be the quotient norm of $f+\mathcal{I}_a^{(n)}$ in $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}$, and put

$$|||f+\mathcal{I}_a^{(n)}|||:=\int_0^a |f^{(n)}(x)|x^n\,dx+\max_{0\le i\le n-1}\{|f^{(i)}(a)|\},$$

$$[[f+\mathcal{I}_a^{(n)}]]:=\int_0^a |f^{(n)}(x)|x^n\,dx\ +\sum_{i=0}^{n-1}\|x^{i+1}f^{(i)}(x)\|_{(0,a]},$$

where $\|x^{i+1}f^{(i)}(x)\|_{(0,a]}:=\sup_{0<x\le a}|x^{i+1}f^{(i)}(x)|$.

Theorem 3.2.6. *The (nonlinear) functionals $|||\cdot|||$ and $[[\cdot]]$ are both well defined on the quotient algebra $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}$. Moreover, $\|\cdot\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}}$, $|||\cdot|||$ and $[[\cdot]]$ are equivalent norms on $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}$.*

Proof. It is clearly sufficient to show that the functionals of the statement are equivalent. We first start with $\|\cdot\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}}$ and $|||\cdot|||$. For $f\in\mathcal{T}^{(n)}(t^n)$ and $h\in\mathcal{I}_a^{(n)}$, we have

$$\begin{aligned}\|f-h\|_{\mathcal{T}^{(n)}(t^n)} &= \int_0^\infty |(f-h)^{(n)}(x)|x^n\,dx\\ &= \int_0^a |f^{(n)}(x)|x^n\,dx+\int_a^\infty |(f-h)^{(n)}(x)|x^n\,dx\\ &\ge \int_0^a |f^{(n)}(x)|x^n\,dx+C_{n,a}\max_{0\le k\le n-1}|f^{(k)}(a)|\end{aligned}$$

by (1.13). Hence,

$$\|f+\mathcal{I}_a^{(n)}\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}}:=\inf_{h\in\mathcal{I}_a^{(n)}}\left\{\|f-h\|_{\mathcal{T}^{(n)}(t^n)}\right\}\ge C_{n,a}|||f+\mathcal{I}_a^{(n)}|||.$$

This shows in particular that $|||f+\mathcal{I}_a^{(n)}|||$ is well defined on $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}$.

For the converse inequality, take $g(x):=\begin{cases} f(x), & x\in(0,a]\\ p(x), & x\in[a,a+1]\\ 0, & x\in[a+1,\infty)\end{cases}$, where

$$p(x)=c_{2n-1}(a+1-x)^{2n-1}+c_{2n-2}(a+1-x)^{2n-2}+\cdots+c_n(a+1-x)^n$$

and $p^{(i)}(a) = f^{(i)}(a)$ for $i = 0, \cdots, n-1$. The polynomial p exists and is unique since its coefficients are the solutions of the Hermite problem of $n \times n$ linear equations

$$\left.\begin{aligned} c_{2n-1} + c_{2n-2} + \cdots + c_n &= f(a) \\ -c_{2n-1}(2n-1) - c_{2n-2}(2n-2) - \cdots - c_n n &= f'(a) \\ \cdots &= \cdots \\ (-1)^{n+1} c_{2n-1}(2n-1)(2n-2)\cdots(n+1) + \cdots + (-1)^{n+1} c_n n! &= f^{(n-1)}(a) \end{aligned}\right\}$$

for which the matrix

$$A_n := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -(2n-1) & -(2n-2) & \cdots & -n \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}(2n-1)\cdots(n+1) & (-1)^{n+1}(2n-2)\cdots n & \cdots & (-1)^{n+1} n! \end{pmatrix}$$

is invertible. In fact, it is readily seen by induction that

$$|A_n| = \prod_{k=1}^{n} (k-1)! \neq 0.$$

It is straightforward to check that $g \in \mathcal{T}^{(n)}(t^n)$. Now, since $f|_{(0,a)} \equiv g|_{(0,a)}$,

$$\begin{aligned} \|f + \mathcal{I}_a^{(n)}\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}} = \|g + \mathcal{I}_a\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}} \; &\leq \; \|g\|_{\mathcal{T}^{(n)}(t^n)} = \int_0^\infty |g^{(n)}(x)| x^n \, dx \\ &= \int_0^a |f^{(n)}(x)| x^n \, dx + \int_a^{a+1} |p^{(n)}(x)| x^n \, dx. \end{aligned}$$

To estimate the second integral we use that the expression for the n-th derivative of p is

$$\begin{aligned} p^{(n)}(x) = {} & (-1)^n c_{2n-1}(2n-1)(2n-2)\cdots(n+1)n(a+1-x)^{n-1} \\ & + (-1)^n c_{2n-2}(2n-2)(2n-3)\cdots n(n-1)(a+1-x)^{n-2} \\ & + \cdots \; + \; (-1)^n c_n n! \end{aligned}$$

so if $x \in (a, a+1)$ we have

$$\begin{aligned} |p^{(n)}(x)| \leq {} & \max_{n \leq i \leq 2n-1} \{|c_i|\} (2n-1)(2n-2)\cdots(n+1)n \cdot \\ & \qquad \cdot [(a+1-x)^{n-1} + (a+1-x)^{n-2} + \cdots + 1] \\ \leq {} & 2^n n^n n \max_{n \leq i \leq 2n-1} \{|c_i|\} = C_n \max_{n \leq i \leq 2n-1} \{|c_i|\}. \end{aligned}$$

On the other hand, the coefficients c_i are linear combinations of the images $f^{(j)}(a)$, because of Cramer's rule

$$\begin{aligned} c_{2n-i} &= |A_n|^{-1} \begin{vmatrix} 1 & \cdots & f(a) & \cdots & 1 \\ -(2n-1) & \cdots & f'(a) & \cdots & -n \\ \vdots & & \vdots & & \vdots \\ (-1)^{n+1}(2n-1)\cdots(n+1) & \cdots & f^{(n-1)}(a) & \cdots & (-1)^{n+1}n! \end{vmatrix} \\ &= |A_n|^{-1}\left(\mathrm{Cof}_{1,i}\cdot f(a) + \mathrm{Cof}_{2,i}\cdot f'(a) + \cdots + \mathrm{Cof}_{n,i}\cdot f^{(n-1)}(a)\right) \\ &= b_{i,0}f(a) + b_{i,1}f'(a) + \cdots + b_{i,n-1}f^{(n-1)}(a), \end{aligned}$$

where the column of the $f^{(j)}(a)$ is the i-th, $\mathrm{Cof}_{k,i}$ is the (k,i) cofactor and $b_{i,j} := |A_n|^{-1}\mathrm{Cof}_{j+1,i}$. Note that the $b_{i,j}$ only depend on n. Hence,

$$\max_{n\le i\le 2n-1}\{|c_i|\} \le C_n \max_{0\le i\le n-1}\{|f^{(i)}(a)|\},$$

and then

$$\begin{aligned} \int_a^{a+1} |p^{(n)}(x)|x^n\,dx &\le C_n \max_{0\le i\le n-1}\{|f^{(i)}(a)|\}\int_a^{a+1} x^n\,dx \\ &= C_{n,a}\max_{0\le i\le n-1}\{|f^{(i)}(a)|\}. \end{aligned}$$

In this way, we have obtained that

$$\begin{aligned} \|f+\mathcal{I}_a\|_{\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}} &\le \int_0^a |f^{(n)}(x)|x^n\,dx + C_{n,a}\max_{0\le i\le n-1}\{|f^{(i)}(a)|\} \\ &\le C|||f+\mathcal{I}_a^{(n)}|||, \end{aligned}$$

as required.

Finally, notice that the inequality $\|x^{k+1}f^{(k)}\|_\infty \le C_n\|f\|_{\mathcal{T}^{(n)}(t^n)}$ $(k = 0,1,\cdots,n-1)$, and Remark 3.2.4 imply that

$$C_{n,a}|||f+\mathcal{I}_a^{(n)}||| \le [[f+\mathcal{I}_a^{(n)}]] \le C_{n,a}|||f+\mathcal{I}_a^{(n)}|||.$$

This concludes the proof. □

Let $\mathcal{V}^{(n)}(0,a)$ denote the space of functions $f\colon (0,a] \to \mathbb{C}$ such that there exist $f',\cdots,f^{(n-1)}$ on $(0,a]$, the function $f^{(n-1)}$ is absolutely continuous on $(0,a]$, and

$$\int_0^a |f^{(n)}(x)|x^n\,dx < \infty.$$

Corollary 3.2.7. *The space $\mathcal{V}^{(n)}(0,a)$, endowed with the convolution product*

$$(f*g)(x) = \int_0^x f(x-y)g(y)\,dy, \qquad x \in (0,a], \quad f,g \in \mathcal{V}^{(n)}(0,a),$$

and the norm

$$|||f||| = \int_0^a |f^{(n)}(x)|x^n\,dx \ + \max_{0\le i\le n-1} |f^{(i)}(a)|, \quad f \in \mathcal{V}^{(n)}(0,a),$$

is a radical Banach algebra isomorphic to $\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)}$.

Proof. By Theorem 3.2.6 we have the isomorphism

$$\mathcal{T}^{(n)}(t^n)/\mathcal{I}_a^{(n)} \cong \mathcal{V}^{(n)}(0,a).$$

as Banach spaces and algebras. Then the result follows by Theorem 3.2.2. □

We call the radical Banach algebra $\mathcal{V}^{(n)}(0,a)$ the *Volterra algebra of absolutely continuous functions of order* n on $(0,a]$, or *Sobolev-Volterra algebra* for short. From now on, we denote by $\|\cdot\|_{\mathcal{V}^{(n)}(0,a)}$ the previous norm $|||\cdot|||$ on $\mathcal{V}^{(n)}(0,a)$.

Note that for $n=0$ we have that $\mathcal{V}^{(n)}(0,a) := L^1(0,a)$, with the norm in this case just presenting the integral part. For $n>0$ and $f \in \mathcal{V}^{(n)}(0,a)$, the norm

$$\begin{aligned}\|f\|_{\mathcal{V}^{(n)}(0,a)} &= \int_0^a |f^{(n)}(x)|x^n\,dx \ + \sup_{0\le k\le n-1} |f^{(k)}(a)| \\ &\sim \int_0^a |f^{(n)}(x)|x^n\,dx \ + \sum_{k=0}^{n-1} \|x^{k+1}f^{(k)}(x)\|_{(0,a]},\end{aligned}$$

is a mixture of L^1 norm and sup-norm. Indeed, the projection

$$p\colon f \mapsto (f(a),\ldots,f^{(n-1)}(a))$$

yields a direct sum decomposition $\mathcal{V}^{(n)}(0,a) = \ker \mathrm{p} \oplus \mathbb{C}^{\mathrm{n}}$, through which the norm $\|\cdot\|_{\mathcal{V}^{(n)}(0,a)}$ becomes the standard coordinatewise topology on $\mathbb{C}^n$ and just the L^1 norm type $\int_0^a |f^{(n)}(x)|x^n\,dx$ on $\ker \mathrm{p}$.

Next, we state some automatic properties of Sobolev-Volterra algebras as regarding discontinuous homomorphisms and Esterle's classification of radical Banach algebras.

(1) Since $\mathcal{V}^{(n)}(0,a)$ is radical with bounded approximate identities it belongs to class 8 defined in [E3], without lying in class 9; i. e., there is no analytic semigroup in $\mathcal{V}^{(n)}(0,a)$ which is bounded on $\{|z|<1, \mathrm{Re}\, z>0\}$. (If it were the case then $L^1_*(0,a)$ would also be in class 9 because $\mathcal{V}^{(n)}(0,a) \hookrightarrow L^1_*(0,a)$, but $L^1_*(0,a)$ cannot belong to class 9 by [CG, Corollary 1].)

In the following three points we assume the continuum hypothesis.

(2) Once again notice that $\mathcal{V}^{(n)}(0,a)$ is radical with bounded approximate identities. Then it contains a copy of $L^1(e^{-t^2})$, see [E1, Theorem 5.1].

(3) Since $\mathcal{V}^{(n)}(0,a)$ is separable as well, we have that there exists a discontinuous homomorphism $\mathcal{V}^{(n)}(0,a) \to L^1_*(0,a)$, see [E1, Corollary 6.6].

(4) Let $\mathbb{C}[[X]]$ denote the algebra of complex formal series in one variable X. Since $\mathcal{V}^{(n)}(0,a)$ is in class 8 it is also in class 5 (see [E3]), so that there exists a one-to-one homomorphism $\mathbb{C}[[X]] \to \mathcal{V}^{(n)}_1(0,a) := \mathcal{V}^{(n)}(0,a) \oplus \mathbb{C}$. Equivalently, there exists a discontinuous homomorphism $C(K) \to \mathcal{V}^{(n)}_1(0,a)$. Furthermore, there is a discontinuous homomorphism $A \to \mathcal{V}^{(n)}_1(0,a)$ for every unital commutative separable Banach algebra A. In particular we can take $A = C^{(m)}[0,a]$ for all $m \in \mathbb{N}$. See [E2, Theorems 6.4 and 6.5] and [E3, pp. 59, 60] as a basis for the above results.

Remark 3.2.8. Similarly to the integer case, we would like to have a representation of the quotient radical Banach algebra $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}^{(\nu)}_a$, which only took into account the behaviour of the functions on the interval $(0,a]$; that is, to have a Volterra-type algebra on $(0,a]$ formed by absolutely continuous functions of *fractional* order ν on $(0,a]$. To obtain such an algebra it sounds sensible to search for an equivalent norm in $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}^{(\nu)}_a$ given in terms of the restriction of $W^\nu f$ on $(0,a]$, like for example, say,

$$\|f + \mathcal{I}^{(\nu)}_a\|_{(\nu),a} := \int_0^a |W^\nu f(x)| x^\nu \, dx + \sup_{0\le\mu\le\nu-1} \{|W^\mu f(a)|\},$$

or something else related with it.

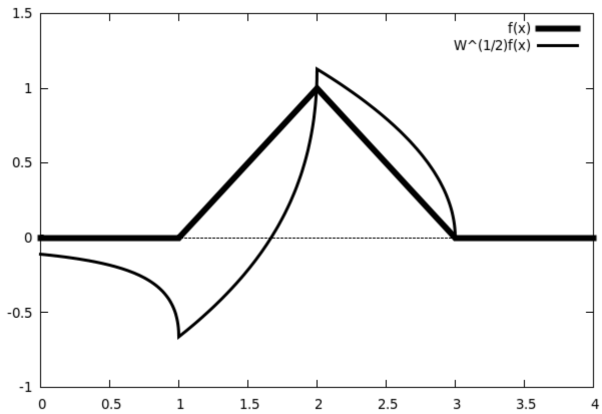
However, it does not work. In fact this question must face a serious obstacle; namely, whereas $\mathcal{I}^{(\nu)}_a$ is invariant under usual derivation, that is, $W^n(\mathcal{I}^{(n)}_a) \subseteq \mathcal{I}^{(n)}_a$, this does not hold for fractional ν. For example, let

$$f(x) := \begin{cases} 0, & x \in (0,a], \\ x-a, & x \in [a,a+1], \\ -x+a+2, & x \in [a+1,a+2], \\ 0, & x \in [a+2,\infty). \end{cases}$$

We have that $f \in \mathcal{T}^{(1)}(t)$, and then $f \in \mathcal{T}^{(\nu)}(t^\nu)$ for all $0<\nu<1$. Moreover, for $0<\nu<1$ and $0<x<a$,

$$\begin{aligned} W^\nu f(x) &= \frac{-1}{\Gamma(1-\nu)} \int_x^\infty (y-x)^{-\nu} f'(y)\, dy \\ &= \frac{-1}{\Gamma(1-\nu)} \int_a^{a+1} (y-x)^{-\nu}\, dy + \frac{1}{\Gamma(1-\nu)} \int_{a+1}^{a+2} (y-x)^{-\nu}\, dy \\ &= \frac{1}{\Gamma(2-\nu)} ((a-x)^{1-\nu} - 2(a+1-x)^{1-\nu} + (a+2-x)^{1-\nu}), \end{aligned}$$

which means that, while $f|_{(0,a]} \equiv 0$ the derivative $W^\nu f$ is such that $W^\nu f|_{(0,a]} \not\equiv 0$ a. e. In other words, $W^\nu(\mathcal{I}_a^{(\nu)}) \not\subseteq \mathcal{I}_a^{(\nu)}$. See the picture (for the case $\nu = 1/2$ and $a = 1$).



QUESTION: Is it possible to characterize the elements of the algebra $\mathcal{T}^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ intrinsically as functions on $(0, a]$?

3.2.3 Closed ideals and derivations of the Sobolev algebra

We are going to show that the standard ideals $\mathcal{I}_x^{(n)}$, $0 \le x \le a$, are the only closed ideals of $\mathcal{V}^{(n)}(0, a)$. Then, because of this and a result of [JS], it follows that all derivations $D\colon \mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$ are automatically continuous. Recall that such a derivation is by definition a linear map such that $D(f * g) = f * D(g) + D(f) * g$, for $f, g \in \mathcal{V}^{(n)}(0, a)$.

Proposition 3.2.9. *Each closed ideal of $\mathcal{V}^{(n)}(0, a)$ is standard.*

Proof. Let I be a closed ideal of $\mathcal{V}^{(n)}(0, a)$. Then $J := q^{-1}(I)$ is a closed ideal of $\mathcal{T}^{(1)}(t)$, where $q\colon \mathcal{T}^{(1)}(t) \to \mathcal{V}^{(n)}(0, a)$ is the canonical quotient mapping. Let $h(J)$ be the hull, or zero-set, of the ideal J in the Gelfand spectrum of $\mathcal{T}^{(1)}(t)$. Any $\xi \in h(J)$ is a character of $\mathcal{T}^{(1)}(t)$ such that $\xi(J) = 0$. Since $q(\mathcal{I}_a^{(n)}) = (0) \subseteq I$ we have that $\mathcal{I}_a^{(n)} \subseteq J$. Hence there is a character $\tilde{\xi}\colon \mathcal{V}^{(n)}(0, a) \equiv \mathcal{T}^{(1)}(t)/\mathcal{I}_a^{(n)} \to \mathbb{C}$ with $\xi = \tilde{\xi} \circ q$. As $\mathcal{V}^{(n)}(0, a)$ is radical it must be the case that $\tilde{\xi} = 0$, and thus $\xi = 0$. In conclusion, $h(J) = \emptyset$. This implies by [GW2, Theorem 3.2] that J is standard in $\mathcal{T}^{(1)}(t)/\mathcal{I}_a^{(n)}$; that is, $J = I_x$ for some $x \in [0, \infty)$. If $x \ge a$ then $I = q(J) = (0)$; if $0 \le x < a$ then $I = q(J) = \mathcal{I}_x^{(n)}$ as required. $\square$

Corollary 3.2.10. *Epimorphisms from Banach algebras onto $\mathcal{V}^{(n)}(0, a)$ and derivations $\mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$ are continuous.*

Proof. Since every closed ideal of $\mathcal{V}^{(n)}(0, a)$ is standard, it is enough to apply [JS, Theorem 2] with an argument similar to that one of [JS, Corollary 4], just using test functions $\varphi \in C_c^{(n)}(0, a)$ instead characteristic (indicator) functions. $\square$

We would like to find a characterization of all derivations from $\mathcal{V}^{(n)}(0,a)$ into itself, such as it has been done for the Volterra algebra $L^1_*(0,a)$ in [KS]. Unfortunately, it is not clear to us how to elucidate that question completely. We give some partial results.

Let $D : \mathcal{V}^{(n)}(0,a) \to \mathcal{V}^{(n)}(0,a)$ be a (bounded) derivation. Let $\mathbf{1}$ denote the constant function $\mathbf{1}(x) = 1$, $x \in (0,a]$. Then, as $\mathbf{1}^{*m} = x^{m-1}/(m-1)!$ we have $Dx^m = m!D(\mathbf{1}^{*(m+1)}) = (m+1)!\mathbf{1}^{*m} * D\mathbf{1} = (m+1)mx^{m-1} * D\mathbf{1}$. Hence, $Dp = (xp)'' * g$ for every polynomial p, with the convention $x'' = \delta_0$ (the Dirac delta at 0), where $g := D\mathbf{1} \in \mathcal{V}^{(n)}(0,a)$. At this point, one can think of getting an expression for the derivation D acting on the (holomorphic) semigroup σ^z defined in Proposition 3.2.5. As before,

$$D(\sigma^z) = (x^z/\Gamma(z))'' * g = z(\sigma^{z-1} * g),$$

so that $\sigma^2 * D(\sigma^z) = z\sigma^{z+1} * g = (x\sigma^z) * g$ whenever $\operatorname{Re} z > 0$. Now the question is to identify the quotient g/σ^2.

By reasoning along the same lines as in [KS] one can try an approximation argument. It is not difficult to see that the above equality for polynomials also holds for test functions f: $Df = (xf)'' * g$, $f \in C_c^{(n+2)}(0,a]$.

Take $(\varphi_m)_{m=1}^\infty \subseteq C_c^{(\infty)}((0,a))$ a bounded approximate identity for $\mathcal{V}^{(n)}(0,a)$ and put $g_m := g * \varphi_m$. Then $g_m \in C^\infty((0,a))$ and $g_m'' = g * \varphi_m''$. Moreover, integration by parts gives us that $(xf)'' * g_m = xf * g_m''$ for all m. Since one may assume that $g_m \to g$ a.e., it is to be expected that g_m'' should converge in some suitable way to a certain measure or distribution μ on $(0,a)$ in analogy with the case $n = 0$; see [KS]. (Then we would have $g = \mu * \sigma^2$.) However, by following an argument similar to that one of [KS], one gets a gap caused by the fact that the algebra $\mathcal{V}^{(n)}(0,a)$ *is not* invariant under right translations.

In the opposite direction, we have the following results.

Lemma 3.2.11. *The application*

$$\begin{aligned} d: \mathcal{V}^{(n)}(0,a) &\to \mathcal{V}^{(n)}(0,a) \\ f(x) &\mapsto xf(x) \end{aligned}$$

is a (bounded) derivation on $\mathcal{V}^{(n)}(0,a)$.

Proof. Obviously, if $f \in \mathcal{V}^{(n)}(0,a)$ then $d(f)$ is absolutely continuous of order n, and $d(f) \in \mathcal{V}^{(n)}(0,a)$ as well. Thus the mapping d is well defined.

Finally, d satisfies the derivation rule. Given $x \in (0,a)$,

$$\begin{aligned} d(f*g)(x) = x(f*g)(x) &= \int_0^x (x-t+t)f(x-t)g(t)\,dt \\ &= \int_0^x (x-t)f(x-t)g(t)\,dt + \int_0^x f(x-t)\,tg(t)\,dt \\ &= (df*g + f*dg)(x). \end{aligned}$$

This concludes the proof. □

From now on, $(tf)^{(k)}(u)$ is used to denote the value at u of the function

$$t \mapsto \frac{d^k(tf(t))}{dt^k}(t), \quad t > 0.$$

The next lemma is needed in order to prove the main result of this section.

Lemma 3.2.12. (1) *Let $k \geq 1$ and $0 < u < a$. Then*

$$(tf)^{(k)}(u) = kf^{(k-1)}(u) + uf^{(k)}(u), \quad \forall f \in \mathcal{V}^{(k)}(0,a).$$

(2) *Let $k \geq 1$, $l \in \{0, \dots, k-1\}$ and $0 < b < a$. Then*

$$\int_0^b y^l (tf)^{(k)}(y)\, dy = \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} b^{l-j} (tf)^{(k-1-j)}(b)$$

for all $f \in \mathcal{V}^{(k)}(0,a)$ null near 0.

(3) *Let $m \geq 0$, $n \in \{0, \dots, m\}$ and $0 < u < a$. Then*

$$u^n (tf)^{(m)}(u) = \left(t \sum_{j=0}^{n} c_{j,n,m} t^j f^{(j)} \right)^{(m-n)} (u), \quad f \in \mathcal{V}^{(k)}(0,a),$$

for certain coefficients $c_{j,n,m} \in \mathbb{R}$.

Proof. We proceed by induction.

(1) The case $k = 1$ and the inductive step $(k) \Rightarrow (k+1)$ are straightforward.

(2) The case $l = 0$ is trivial for all $k \geq 1$. The inductive step $(k-1, l) \Rightarrow (k, l+1)$, for $k \geq 2$, is as follows:

$$\begin{aligned}
\int_0^b y^{l+1} (tf)^k(y)\, dy &= b^{l+1} (tf)^{(k-1)}(b) - (l+1) \int_0^b y^l (tf)^{(k-1)}(y)\, dy \\
&= b^{l+1} (tf)^{(k-1)}(b) - (l+1) \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} b^{l-j} (tf)^{(k-2-j)}(b) \\
&= \sum_{m=0}^{l+1} (-1)^m \frac{(l+1)!}{(l+1-m)!} b^{l+1-m} (tf)^{(k-1-m)}(b),
\end{aligned}$$

where we have integrated by parts and applied the induction hypothesis at level $(k-1, l)$.

(3) The case $n = 0$ is trivial for all $m \geq 0$, with $c_{0,0,m} = 1$. Also the case $n = m$, with $m \geq 1$ is straightforward, by using part (i), with $c_{j,m,m} = 0$ for $j \in \{0, \dots, n-2\}$ (if $m \geq 2$), $c_{m-1,m,m} = m$ and $c_{m,m,m} = 1$. Now the inductive step is

$$\left. \begin{array}{l} (m, n-1) \\ (m, n) \end{array} \right\} \Rightarrow (m+1, n)$$

for $m \geq n \geq 1$. Then

$$\begin{aligned} u^n(tf)^{(m+1)}(u) &= \left(t^n(tf)^{(m)}\right)'(u) - nu^{n-1}(tf)^{(m)}(u) \\ &= \left(\left(t\sum_{j=0}^{n} c_{j,n,m}t^j f^{(j)}\right)^{(m-n)}\right)'(u) - n\left(t\sum_{j=0}^{n-1} c_{j,n-1,m}t^j f^{(j)}\right)^{(m-(n-1))}(u) \\ &= \left(t\sum_{j=0}^{n} c_{j,n,m+1}t^j f^{(j)}\right)^{(m+1-n)}(u) \end{aligned}$$

with $c_{n,n,m+1} = c_{n,n,m}$, (and therefore $c_{n,n,m+1} = c_{n,n,m} = \ldots = c_{n,n,n} = 1$), and

$$c_{j,n,m+1} = c_{j,n,m} - nc_{j,n-1,m}, \quad \text{for } j = 0, \ldots, n-1.$$

This concludes the proof. □

Remark 3.2.13. As a matter of fact, the coefficients $c_{j,n,m}$ are the following:
If $m = n = 0$,

$$c_{0,0,0} = 1.$$

If $m = n \geq 1$,

$$c_{n,n,n} = 1, \quad c_{n-1,n,n} = n \quad \text{and } c_{j,n,n} = 0 \text{ for } j = 0, \ldots, n-2, \text{ if } n \geq 2.$$

If $m = n + 1 \geq 2$,

$$c_{n,n,n+1} = 1 \quad \text{and } c_{j,n,n+1} = 0 \text{ for } j = 0, \ldots, n-1.$$

Finally, if $m - n \geq 2$,

$$c_{j,n,m} = (-1)^{n-j}\binom{n}{j}\frac{(m-2-j)!}{(m-2-n)!}, \quad \text{for } j = 0, \ldots, n.$$

We have not included the value of the coefficients in the formulation of the lemma in order not to make too long the statement (and the proof of it). Note that the exact expression of the coefficients is not important to establish the estimates.

Through the proof of the following theorem we will assume that $f \in C^{(n)}(0, a)$ and it vanishes near the origin. Then for a continuous function μ on $[0, a)$, there exists the function defined on $[0, a)$ given by the convolution $xf * \mu$ and it is derivable up to the order n on $[0, a)$, with

$$(xf * \mu)^{(j)} = (xf)^{(j)} * \mu, \quad \text{for each } j = 0, \ldots, n.$$

As usual we will identify $d\mu_j(t)$ and $\mu_j(t)\,dt$ when necessary.

Theorem 3.2.14. *Fix $n \geq 1$. Let $\mu_0, \ldots, \mu_{n-1}$ be n derivable functions on $[0,a)$, and let μ_n be a Borel measure on $[0,a)$ satisfying*

(a)
$$\sup_{0<s<a} s \int_0^{a-s} |d\mu_j|(t) < \infty \quad (j = 0, \ldots, n),$$

(b)
$$\int_0^s d\mu_{j+1}(t) = s\mu_j(s) - (j+1) \int_0^s \mu_j(t)\, dt \quad (s \in [0,a); 0 \leq j \leq n-1).$$

Then

(1) *For each $k = 0, \ldots, n$ and $j \in \{0, \ldots, n-k\}$,*
$$\int_0^a |(xf * \mu_j)^{(k)}(x)| x^k\, dx \leq C\|f\|_{\mathcal{V}^{(k)}(0,a)}, \quad \forall f \in \mathcal{V}^{(k)}(0,a).$$

(2) *For each $k = 1, \ldots, n$ and $j \in \{0, \ldots, n-k\}$,*
$$\|x^k (xf * \mu_j)^{(k-1)}(x)\|_{(0,a]} \leq C\|f\|_{\mathcal{V}^{(k)}(0,a)}, \quad \forall f \in \mathcal{V}^{(k)}(0,a).$$

In consequence, the operators $f \mapsto xf * \mu_j$, $j = 0, \ldots, n-k$, are bounded from $\mathcal{V}^{(k)}(0,a)$ to $\mathcal{V}^{(k)}(0,a)$ for each $k = 0, \ldots, n$.

Proof. We proceed by induction on k. By density, it will be enough to prove the inequalities for functions $f \in C^{(k)}(0,a)$ null near the origin.

(1) The case $k = 0$ is given in [KS], but we include it here for the convenience of the reader. Let $j \in \{0, \ldots, n\}$. Then

$$\begin{aligned}
\|xf * \mu_j\|_1 &\leq \int_0^a \int_0^x (x-t)|f(x-t)||d\mu_j|(t)\, dx \;=\; \int_0^a \int_0^{a-t} s|f(s)| ds |d\mu_j|(t) \\
&= \int_0^a |f(s)| \left(s \int_0^{a-s} |d\mu_j|(t) \right) ds \;\leq\; C\|f\|_1
\end{aligned}$$

where we have applied condition (a). Now let $k \in \{1, \ldots, n\}$ and suppose that the statement is true for $0, 1, \ldots, k-1$. Let $f \in \mathcal{V}^{(k)}(0,a)$ be null near to 0, and $j \in \{0, \ldots, n-k\}$. Then

$$\int_0^a |(xf * \mu_j)^{(k)}(x)| x^k\, dx \leq I_{k,1} + I_{k,2},$$

where
$$I_{k,1} := \int_0^a \Big| \int_0^x (tf)^{(k)}(y)(x^k - y^k)\mu_j(x-y)\, dy \Big|\, dx,$$

$$I_{k,2} := \int_0^a \Big| \int_0^x (tf)^{(k)}(y)\mu_j(x-y)y^k\,dy\Big|\,dx.$$

Now, to estimate the first integral $I_{k,1}$, notice that applying the cyclotomic identity

$$x^k - y^k = (x-y)\sum_{l=0}^{k-1} x^{k-1-l}y^l,$$

condition (b), and Fubini's theorem, give us

$$\begin{aligned}
I_{k,1} &= \int_0^a \Big| \int_0^x \left(\sum_{l=0}^{k-1} x^{k-1-l}y^l\right)(tf)^{(k)}(y)\left(\int_0^{x-y} d\mu_{j+1}(s) + (j+1)\int_0^{x-y} d\mu_j(s)\right)dy\Big|\,dx \\
&= \int_0^a \Big| \int_0^x \left(\int_0^{x-s}\left(\sum_{l=0}^{k-1} x^{k-1-l}y^l\right)(tf)^{(k)}(y)\,dy\right)(d\mu_{j+1}(s) + (j+1)d\mu_j(s))\Big|\,dx.
\end{aligned}$$

From now on, to simplify our notation, denote

$$\mu_\bullet := \mu_{j+1} + (j+1)\mu_j, \quad \text{so} \quad d\mu_\bullet(s) := d\mu_{j+1}(s) + (j+1)d\mu_j(s).$$

We can use Lemma 3.2.12 (2) and (3) to get

$$\begin{aligned}
I_{k,1} &= \int_0^a \Big| \sum_{l=0}^{k-1} x^{k-1-l}\int_0^x \sum_{j=0}^{l}(-1)^j\frac{l!}{(l-j)!}(x-s)^{l-j}(tf)^{(k-1-j)}(x-s)d\mu_\bullet(s)\Big|\,dx \\
&= \int_0^a \Big| \sum_{l=0}^{k-1}\sum_{j=0}^{l}(-1)^j\frac{l!}{(l-j)!}x^{k-1-l}\left(t^{l-j}(tf)^{(k-1-j)} * \mu_\bullet\right)(x)\Big|\,dx \\
&= \int_0^a \left| \sum_{l=0}^{k-1}\sum_{j=0}^{l}(-1)^j\frac{l!}{(l-j)!}x^{k-1-l}\left(\left(t\sum_{m=0}^{l-j} C_{m,l,j,k}t^m f^{(m)}\right)^{(k-1-l)} * \mu_\bullet\right)(x)\right|\,dx.
\end{aligned}$$

By the comment prior to the statement of the theorem, the $(k-1-l)$-th derivative affects to the whole convolution product, therefore

$$\begin{aligned}
I_{k,1} &\le C_k\sum_{l=0}^{k-1}\sum_{j=0}^{l}\|\left(t\sum_{m=0}^{l-j} C_{m,l,j,k}t^m f^{(m)}\right) * \mu_\bullet\|_{\mathcal{V}^{(k-1-l)}(0,a)} \\
&\le C_k\sum_{l=0}^{k-1}\sum_{j=0}^{l}\|\sum_{m=0}^{l-j} C_{m,l,j,k}t^m f^{(m)}\|_{\mathcal{V}^{(k-1-l)}(0,a)} \;\le\; C_k\sum_{l=0}^{k-1}\sum_{j=0}^{l}\sum_{m=0}^{l-j}\|t^m f^{(m)}\|_{\mathcal{V}^{(k-1-l)}(0,a)} \\
&\le C_k\sum_{l=0}^{k-1}\sum_{m=0}^{l}\|t^m f^{(m)}\|_{\mathcal{V}^{(k-1-l)}(0,a)}.
\end{aligned}$$

Here we have applied the induction hypothesis over μ_j and μ_{j+1} at levels $0, 1, \ldots, k-1$. By Remark 3.2.4, we have the continuous inclusions

$$\mathcal{V}^{(k)}(0,a) \hookrightarrow \mathcal{V}^{(k-1)}(0,a) \hookrightarrow \cdots \hookrightarrow \mathcal{V}^{(0)}(0,a) = L^1(0,a),$$

so to get the bound for $I_{k,1}$ it suffices to prove that

$$\|t^j f^{(j)}\|_{\mathcal{V}^{(k-1-j)}} \le C_k \|f\|_{\mathcal{V}^{(k-1)}}, \quad \text{for all } j = 0, \ldots, k-1.$$

This is just a direct calculation:

$$\begin{aligned}
\|t^j f^{(j)}\|_{\mathcal{V}^{(k-1-j)}} &= \int_0^a \Big| \left(t^j f^{(j)}\right)^{(k-1-j)} (u) \Big| u^{k-1-j} du \\
&= \int_0^a \Big| \sum_{m=0}^{k-1-j} \binom{k-1-j}{m} \left(t^j\right)^{(m)} (u) \left(f^{(j)}\right)^{(k-1-j-m)} (u) \Big| u^{k-1-j} du \\
&= \int_0^a \Big| \sum_{m=0}^{\min\{j,k-1-j\}} \binom{k-1-j}{m} \frac{j!}{(j-m)!} u^{j-m} f^{(k-1-m)}(u) \Big| u^{k-1-j} du \\
&\le C_k \sum_{m=0}^{\min\{j,k-1-j\}} \|f\|_{\mathcal{V}^{(k-1-m)}(0,a)} \;\le\; C_k \|f\|_{\mathcal{V}^{(k-1)}(0,a)}.
\end{aligned}$$

As regards the second integral, $I_{k,2}$, one has

$$\begin{aligned}
I_{k,2} &\le \int_0^a \int_0^x (y^k |f^{(k)}(y)| + k y^{k-1} |f^{(k-1)}(y)|) y |\mu_j(x-y)| \, dy \, dx \\
&= \int_0^a (y^k |f^{(k)}(y)| + k y^{k-1} |f^{(k-1)}(y)|) y \int_0^{a-y} |\mu_j(s)| \, ds \, dy \\
&\le C \left(\|f\|_{\mathcal{V}^{(k)}(0,a)} + k \|f\|_{\mathcal{V}^{(k-1)}(0,a)} \right) \le C_{n,a} \|f\|_{\mathcal{V}^{(k)}(0,a)}.
\end{aligned}$$

Here we have applied Lemma 3.2.12 (1), Fubini's theorem, condition (a) and Remark 3.2.4.

(2) In the base case $k = 1$, for $j \in \{0, \ldots, n-1\}$, we have

$$\|x(xf * \mu_j)(x)\|_{(0,a]} \le J_{1,1} + J_{1,2},$$

with

$$J_{1,1} := \sup_{0<x<a} \Big| \int_0^x (x-y) \mu_j(x-y) y f(y) \, dy \Big|,$$

and

$$J_{1,2} := \sup_{0<x<a} \Big| \int_0^x y^2 \mu_j(x-y) f(y) \, dy \Big|.$$

For the first supremum,

$$\begin{aligned}J_{1,1} &\leq \sup_{0<x<a}\left(\int_0^x |f(y)|\left(y\int_0^{x-y}|d\mu_{j+1}|(s)+(j+1)y\int_0^{x-y}|d\mu_j|(s)\right)\,dy\right)\\&\leq C\|f\|_1\leq C\|f\|_{\mathcal{V}^{(1)}(0,a)},\end{aligned}$$

where we have applied conditions (b) and (a) and Remark 3.2.4.

For the second supremum,

$$\begin{aligned}J_{1,2} &= \sup_{0<x<a}\left|-\int_0^x\left(\int_0^{x-y}d\mu_j(s)\right)[t^2f(t)]'(y)\,dy\right|\\&\leq \sup_{0<x<a}\left(\int_0^x\left(y\int_0^{x-y}|d\mu_j|(s)\right)(2|f(y)|+y|f'(y)|)\,dy\right)\\&\leq C(2\|f\|_1+\|f\|_{\mathcal{V}^{(1)}(0,a)})\ \leq\ C\|f\|_{\mathcal{V}^{(1)}(0,a)}.\end{aligned}$$

Now take $k\in\{2,\ldots,n\}$ and suppose that the statement is true for $0,1,\ldots,k-1$. For $j\in\{0,\ldots,n-k\}$,

$$\|x^k(xf*\mu_j)^{(k-1)}(x)\|_{(0,a]}\leq J_{k,1}+J_{k,2},$$

where

$$J_{k,1}:=\sup_{0<x<a}\left|\int_0^x(x^k-y^k)\mu_j(x-y)(tf)^{(k-1)}(y)\,dy\right|,$$

and

$$J_{k,2}:=\sup_{0<x<a}\left|\int_0^x y^k\mu_j(x-y)(tf)^{(k-1)}(y)\,dy\right|.$$

With a similar argument to that used to estimate $I_{k,1}$, we get

$$\begin{aligned}J_{k,1} &= \sup_{0<x<a}\left|\int_0^x\sum_{l=0}^{k-1}x^{k-1-l}\left(\int_0^{x-s}y^l(tf)^{(k-1)}(y)\,dy\right)d\mu_\bullet(s)\right|\\&\leq A_{k,1}+B_{k,1},\end{aligned}$$

where

$$A_{k,1}:=\sup_{0<x<a}\left|\int_0^x\sum_{l=0}^{k-2}x^{k-1-l}\left(\int_0^{x-s}y^l(tf)^{(k-1)}(y)\,dy\right)d\mu_\bullet(s)\right|$$

and

$$B_{k,1}:=\sup_{0<x<a}\left|\int_0^x\left(\int_0^{x-s}y^{k-1}(tf)^{(k-1)}(y)\,dy\right)d\mu_\bullet(s)\right|.$$

For $A_{k,1}$, we proceed as we did for $I_{k,1}$,

$$
\begin{aligned}
A_{k,1} &= \sup_{0<x<a} \Big| \int_0^x \sum_{l=0}^{k-2} x^{k-1-l} \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} (x-s)^{l-j} (tf)^{k-2-j} (x-s) d\mu_\bullet(s) \Big| \\
&= C_k \sum_{l=0}^{k-2} \sum_{j=0}^{l} \| x^{k-1-l} \left(\left(t \sum_{i=0}^{l-j} C_{i,j,l,k} t^i f^{(i)} \right) * \mu_\bullet \right)^{(k-2-l)} (x) \|_{(0,a]} \\
&\le C_k \sum_{l=0}^{k-2} \sum_{j=0}^{l} \| \sum_{i=0}^{l-j} C_{i,j,l,k} t^i f^{(i)} \|_{\mathcal{V}^{(k-1-l)}(0,a)} \ \le \ C_k \|f\|_{\mathcal{V}^{(k-1)}(0,a)}.
\end{aligned}
$$

For $B_{k,1}$, we apply Fubini's theorem and get

$$
\begin{aligned}
B_{k,1} &= \sup_{0<x<a} \Big| \int_0^x \left(\int_0^{x-y} d\mu_\bullet(s) \right) y^{k-1} (tf)^{(k-1)}(y)\, dy \\
&\le \sup_{0<x<a} \int_0^x \left(\int_0^{x-y} |d\mu_\bullet|(s) \right) y^{k-1} |(tf)^{(k-1)}(y)|\, dy \\
&\le \sup_{0<x<a} \int_0^x \left(\sup_{0<y<x} y \int_0^{x-y} |d\mu_\bullet|(s) \right) y^{k-2} |(tf)^{(k-1)}(y)|\, dy \\
&\le C((k-1)\|f\|_{\mathcal{V}^{(k-2)}(0,a)} + \|f\|_{\mathcal{V}^{(k-1)}(0,a)}) \le C_k \|f\|_{\mathcal{V}^{(k-1)}(0,a)}.
\end{aligned}
$$

where we have used condition (a), Lemma 3.2.12 (1) and Remark 3.2.4.

Finally, for the second supremum,

$$
\begin{aligned}
J_{k,2} = \sup_{0<x<a} \Big| \int_0^x \int_0^y & d\mu_j(s) (k(x-y)^{k-1} (tf)^{(k-1)}(x-y) \\
& + (x-y)^k (tf)^{(k)}(x-y))\, dy \Big| \\
\le \sup_{0<x<a} (\int_0^x r \int_0^{x-r} & |d\mu_j|(s) (k(k-1) r^{k-2} |f^{(k-2)}(r)| \\
& + 2k r^{k-1} |f^{(k-1)}(r)| + r^k |f^{(k)}(r)|)\, dr) \\
\le C(k(k-1)\|f\|_{\mathcal{V}^{(k-2)}(0,a)} & + 2k\|f\|_{\mathcal{V}^{(k-1)}(0,a)} + \|f\|_{\mathcal{V}^{(k)}(0,a)}) \\
\le C_{n,a} \|f\|_{\mathcal{V}^{(k)}(0,a)}, &
\end{aligned}
$$

where we have applied again condition (a) and Remark 3.2.4.

With all the above estimates, the proof is done. □

From Theorem 3.2.14 one gets immediately the following result.

Corollary 3.2.15. *Take $n \ge 1$. Let $\mu_0, \dots, \mu_{n-1}$ be n derivable functions on $[0,a)$, and let μ_n be a Borel measure on $[0,a)$ satisfying*

(1)
$$\sup_{0<s<a} s \int_0^{a-s} |d\mu_j|(t) < \infty \quad (j = 0, \dots, n),$$

(2)
$$\int_0^s d\mu_{j+1}(t) = s\,\mu_j(s) - (j+1)\int_0^s \mu_j(t)\,dt \quad (s \in [0,a); 0 \le j \le n-1).$$

Then the linear mapping $f \mapsto xf * \mu_0$ is a (bounded) derivation from $\mathcal{V}^{(n)}(0,a)$ to $\mathcal{V}^{(n)}(0,a)$.

QUESTION: Is every derivation $D\colon \mathcal{V}^{(n)}(0,a) \to \mathcal{V}^{(n)}(0,a)$ of the form given in Corollary 3.2.15 ?

If we knew how to describe all the derivations on $\mathcal{V}^{(n)}(0,a)$ then one could pose naturally in this setting the problem of finding the automorphisms of the algebra $\mathcal{V}^{(n)}(0,a)$, and whether or not the group of such automorphisms is connected in the operator norm topology on $\mathcal{V}^{(n)}(0,a)$ (see [Gh] for $n = 0$).

Chapter 4

RKHS and Cesàro-Hardy operators

4.1 Lebesgue-Sobolev spaces as Reproducing Kernel Hilbert Spaces

In this chapter we focus on the case $p = 2$. It is clear that the space $\mathcal{T}_2^{(\nu)}(t^\nu)$ is, for every $\nu > 0$, a Hilbert space with inner product

$$(4.1)\qquad (f|g)_{2,(\nu)} := \int_0^\infty W^\nu f(t)\overline{W^\nu g(t)}t^{2\nu}dt, \quad f, g \in \mathcal{T}_2^{(\nu)}(t^\nu).$$

Via the isometry provided by $\mathcal{C}_\nu^*$, one can find a suitable ortonormal basis in $\mathcal{T}_2^{(\nu)}(t^\nu)$. Let $(\ell_m)_{m=0}^\infty$ be the orthonormal system on $L_2(\mathbb{R}^+)$ of Laguerre functions ℓ_m given by

$$\ell_m(t) = e^{-t/2}\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^j}{j!}t^j, \quad t > 0,\ m = 0, 1, \dots$$

Set $\ell_{m,\nu} := W^{-\nu}(t^{-\nu}\ell_m)$, that is, for $t > 0$,

$$(4.2)\qquad \ell_{m,\nu}(t) = \sum_{j=0}^{m}\binom{k}{j}\frac{(-1)^j}{j!}\frac{1}{\Gamma(\nu)}\int_1^\infty (u-1)^{\nu-1}u^{j-\nu}u^j e^{-ut/2}du.$$

Then $(\ell_{m,\nu})_{m=0}^\infty$ is an orthonormal basis in $\mathcal{T}_2^{(\nu)}(t^\nu)$ since $W^{-\nu}$ is an isometry from $L_2(t^{2\nu})$ onto $\mathcal{T}_2^{(\nu)}(t^\nu)$. This basis will be used in Section 4.3.

As it has been pointed out at the end of Section 1.3 in Chapter 1, for $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$ one has that $f(t)$ exists and $|f(t)| \le t^{-1/2}C\|f\|_{2,(\nu)}$ for every $t > 0$ and $\nu > 1/2$. Thus, $\mathcal{T}_2^{(\nu)}(t^\nu)$ is a reproducing kernel Hilbert space (RKHS for short). Our aim next is to find an expression of the reproducing kernel in $\mathcal{T}_2^{(\nu)}(t^\nu)$.

Let $t > 0$. The space $\mathcal{T}_2^{(\nu)}(t^\nu)$ is a RKHS if and only if there exists $k_{\nu,t} \in \mathcal{T}_2^{(\nu)}(t^\nu)$ such that $(f \mid k_{\nu,t})_{2,(\nu)} = f(t)$ for all $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$. On the other hand,

$$f(t) = \frac{1}{\Gamma(\nu)}\int_t^\infty (u-t)^{\nu-1}W^\nu f(u)du = \frac{1}{\Gamma(\nu)}\int_0^\infty W^\nu f(u)(u-t)_+^{\nu-1}du,$$

and therefore

$$\int_0^\infty W^\nu f(u)\overline{W^\nu k_{\nu,t}(u)}u^{2\nu}du = \int_0^\infty W^\nu f(u)(u-t)_+^{\nu-1}\frac{du}{\Gamma(\nu)}$$

for every f in $\mathcal{T}_2^{(\nu)}(t^\nu)$, so for every $f \in C_c^{(\infty)}(0,\infty)$. It follows then that

$$W^\nu k_{\nu,t}(u) = \frac{1}{\Gamma(\nu)}\frac{(u-t)_+^{\nu-1}}{u^{2\nu}}, \quad t,u>0. \tag{4.3}$$

Note that the last function is in $L_2(u^{2\nu})$ if and only if $\nu > 1/2$. In this case we have

$$\begin{aligned} k_{\nu,t}(s) &= W^{-\nu}\left[W^\nu k_{\nu,t}\right](s) = \frac{1}{\Gamma(\nu)}\int_s^\infty (r-s)^{\nu-1}W^\nu k_{\nu,t}(u)du \\ &= \frac{1}{\Gamma(\nu)^2}\int_0^\infty (r-s)_+^{\nu-1}(r-t)_+^{\nu-1}r^{-2\nu}dr, \quad s>0. \end{aligned}$$

In conclusion, we have proved the following result. Put $k_\nu(s,t) := k_{\nu,t}(s)$.

Proposition 4.1.1. *Let $\nu > 0$. The Hilbert space $\mathcal{T}_2^{(\nu)}(t^\nu)$ is RKHS if and only if $\nu > 1/2$. In this case, the kernel for $\mathcal{T}_2^{(\nu)}(t^\nu)$ is the function*

$$k_\nu(s,t) = \int_0^\infty g_\nu(s,r)g_\nu(t,r)dr, \quad t,s>0, \tag{4.4}$$

where

$$g_\nu(t,r) = \frac{(r-t)_+^{\nu-1}}{r^\nu\Gamma(\nu)}, \quad t,r>0.$$

Remarks 4.1.2. Spaces $\mathcal{T}_2^{(\nu)}(t^\nu)$ can be considered as spaces formed by paths (of infinite length, in this case) as it happens with other typical examples in the theory of reproducing kernels. One can readily describe some standard or general facts of the theory of reproducing kernels in our setting.

(1) Norm of the kernel. For $\nu > 1/2$, the $\mathcal{T}_2^{(\nu)}$-norm of $k_{\nu,t}$ one has

$$\begin{aligned} \|k_{\nu,t}\|_{2,(\nu)}^2 &= k_\nu(t,t) = \frac{1}{\Gamma(\nu)^2}\int_t^\infty (s-t)^{2\nu-2}s^{-2\nu}ds \\ &= \frac{1}{\Gamma(\nu)^2}\int_t^\infty \left(1-\frac{t}{s}\right)^{2\nu-2}s^{-2}ds = \frac{1}{\Gamma(\nu)^2 t}\int_0^1 (1-u)^{2\nu-2}du \\ &= \frac{B(1,2\nu-1)}{\Gamma(\nu)^2}\frac{1}{t} = \frac{1}{\Gamma(\nu)^2(2\nu-1)}\frac{1}{t}. \end{aligned}$$

That is,

$$\|k_{\nu,t}\|_{2,(\nu)} = \frac{1}{\Gamma(\nu)\sqrt{2\nu-1}}\frac{1}{\sqrt{t}}, \quad t>0.$$

(2) Hypergeometric function. The kernel k_ν can be rewritten in terms of the hypergeometric function ${}_2F_1$. Recall that for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|z| < 1$, that function can be expressed by

$$B(b, c-b)_2F_1(a, b, c, z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt.$$

(where B is the beta function). Denote $t \wedge s = \min(t, s)$ and $t \vee s = \max(t, s)$. Then

$$\begin{aligned}
k_\nu(s,t) &= \int_0^\infty \frac{(r-s)_+^{\nu-1}}{r^\nu\Gamma(\nu)} \frac{(r-t)_+^{\nu-1}}{r^\nu\Gamma(\nu)} dr = \int_{t\vee s}^\infty \frac{r^{\nu-1}\left(1-\frac{s}{r}\right)^{\nu-1} r^{\nu-1}\left(1-\frac{t}{r}\right)^{\nu-1}}{\Gamma(\nu)^2 r^{2\nu}} dr \\
&= \int_0^{\frac{1}{s}\wedge\frac{1}{t}} \frac{(1-su)^{\nu-1}(1-tu)^{\nu-1}}{\Gamma(\nu)^2} du \\
&= \int_0^1 \frac{\left(1-\left(1\wedge\frac{s}{t}\right)y\right)^{\nu-1}\left(1-\left(1\wedge\frac{t}{s}\right)y\right)^{\nu-1}}{(s\vee t)\Gamma(\nu)^2} dy \\
&= \frac{1}{(s\vee t)\Gamma(\nu)\Gamma(\nu+1)} {}_2F_1\left(1-\nu, 1, \nu+1, \frac{s\wedge t}{s\vee t}\right)
\end{aligned}$$

When $\nu = n \in \mathbb{N}$ one obtains

$$\begin{aligned}
k_n(s,t) &= \frac{1}{(s\vee t)(n-1)!n!} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{(1)_j}{(n+1)_j} \left(\frac{s\wedge t}{s\vee t}\right)^j \\
&= \sum_{j=0}^{n-1} \frac{(-1)^j}{(n+j)!(n-j-1)!} \frac{(s\wedge t)^j}{(s\vee t)^{j+1}}.
\end{aligned}$$

where $(a)_b$ is the Pochhammer symbol, and the second equality is a simplification.

If $\nu = 1$, we find $k_1(s,t) = \frac{1}{s\vee t} = \frac{1}{s} \wedge \frac{1}{t}$, which reminds us the function $\mathfrak{b}(s,t) = s \wedge t$, that is, the well known reproducing kernel of the RKHS related to Brownian motion (or the covariance of the Brownian process). This suggests to investigate the possible relationship between spaces $\mathcal{T}_2^{(\nu)}(t^\nu)$ and spaces of the Brownian motion. We will consider this item in Section 4.2.

(3) Green function. Given a differential operator L and an equation $Lu(x) = f(x)$, the Green function for L is the solution (whenever it exists) $u = G$ to the twin equation $Lu(x, s) = \delta(x - s)$, where δ is the Dirac delta distribution. Then, once G has been found, one obtains $u(x) = \int G(x, s)f(s)ds$ as solution to the initial equation. From the reproducing kernel theory and from (4.3), we have that g_ν is Green's function for a certain operator L_g. This operator is given in terms of the fractional differential operator W^ν or of the Cesàro-Hardy operator. In effect,

$$g_\nu(t,r) = \frac{(r-t)_+^{\nu-1}}{r^\nu\Gamma(\nu)} \Leftrightarrow r^\nu g_\nu(t,r) = \frac{(r-t)_+^{\nu-1}}{\Gamma(\nu)} \Leftrightarrow W^\nu\left(s^\nu g_\nu(\cdot, s)\right)(t) = \delta_s(t),$$

so that

$$L_h = W^\nu \circ \mu_{-\nu} = \mu_\nu \circ (\mathcal{C}_\nu^*)^{-1} \circ \mu_{-\nu}.$$

(4) Green kernel integral. It is well known that the Green function asociated with a reproducing kernel allows us to recover the Hilbert space that the kernel generates via an integral transformation. In our case such a transformation $\mathfrak{T}$ is, up to constants, the Cesàro-Hardy operator $\mathcal{C}_\nu^*$:

$$\mathfrak{T}f(t) = \int_0^\infty g_\nu(t,r) f(r) dr = \int_t^\infty \frac{(r-t)^{\nu-1}}{\Gamma(\nu) r^\nu} f(r) dr, \quad t>0, f \in L_2(\mathbb{R}^+);$$

that is, $\mathfrak{T} = \Gamma(\nu+1)^{-1}\mathcal{C}_\nu^*$ and so $\mathcal{T}_2^{(\nu)}(t^\nu) = \mathfrak{T}(L_2(\mathbb{R}^+))$, as it had to be !, see [PR, Th. 11.3, Cor. 11.4] or [S2, Th. 1, p.4].

4.2 RKH-Sobolev spaces and Brownian motion

We know (see for example [Lo]) that, for a given definite-positive kernel k, there exists a (unique) Gaussian, zero mean, stochastic process X_t such that the covariance is given by the kernel, which is to say $Cov(X_t, X_s) = k(t,s)$. Let B_t denote the well known Brownian motion, or Wiener process, whose covariance is

$$\mathfrak{b}_0(s,t) = \min\{s,t\} = \int_0^{t\wedge s} \chi_{(0,s)}(u)\chi_{(0,t)}(u) du, \quad s,t>0.$$

The Brownian motion and its main properties can be found in many textbooks, see for example [Du], [KaSh], [P], [SP].

With the aim to provide useful models for the study of random phenomena with a strong interdependence between distant samples, the Brownian motion was widened to fractional Brownian motion (fBm, for short) by B. B. Mandelbrot and J. W. Van Ness in their seminal paper [MV]. The n-times integrated Brownian motion $B_{n,t}$ can be defined recursively,

$$B_{1,t} = \int_0^t B_s ds, \qquad B_{n,t} = \int_0^t B_{n-1,s}\, ds,$$

or explicitly,

$$B_{n,t} = \int_0^t \frac{(t-s)^n}{n!} dB_s,$$

where the previous integrals have to be understood in terms of stochastic integration. ($B_{n,t}$ was first mentionned by Shepp, [Sh, p.327], according to Lachal, see [L] and references therein). For $n=0$ we retrieve the Brownian motion, and the $n=1$ case is usually called the Langevin process [J]. The order of integration need not be a positive integer, in fact P. Lèvy had already introduced in [Le] the Holmgren-Riemann-Liouville fractional integral of B_t, as cited in [MV]. Note that, in the definition of $B_{n,t}$, there is a lack of coordination between the subindex n and the order of integration. A more natural definition, from a mathematical point of view, seems to be the integrated white noise,

$$WN_{\nu,t} := B_{\nu-1,t} = \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} dB_s, \quad \nu > 0.$$

Although the Brownian motion is nowhere differentiable with probability one, the white noise is somehow its formal derivative (see [P, p.140]).

In practice, the kernel (covariance) of the fBm is complicated, apart from the fact that fBm is not suitable for modelling phenomena only arising in positive time, for example. Thus fBm is modified to simplify computations or to deal with specific type of problems, see [BA], [FP], [Hu], [SL] for instance. In the above references, as well as in other works, the tool used to approach questions involving Brownian phenomena is that one of fractional (integral) calculus.

For $\nu > 0$, let $\mathcal{D}^{-\nu}$ be the Riemann-Liouville integral operator

$$\mathcal{D}^{-\nu} f(x) := \int_0^x \frac{(x-y)^{\nu-1}}{\Gamma(\nu)} f(y) dy, \qquad x > 0, \quad f \in L_2(\mathbb{R}^+).$$

Since $\mathcal{D}^{-\nu} G = \mathfrak{r}_\nu * G$, Titchmarsh's convolution theorem implies that $\mathcal{D}^{-\nu}$ is injective. Define $\mathcal{R}_2^{(\nu)} := \mathcal{D}^{-\nu}(L_2(\mathbb{R}^+))$, so that for all $\varphi \in \mathcal{R}_2^{(\nu)}$ there exists a unique $\varphi^{(\nu)} \equiv \mathcal{D}^{-\nu}\varphi$ in $L_2(\mathbb{R}^+)$ such that

$$\varphi(x) = \int_0^x \frac{(x-y)^{\nu-1}}{\Gamma(\nu)} \varphi^{(\nu)}(y) dy, \quad x > 0.$$

Then $\mathcal{R}_2^{(\nu)}$ is endowed with the norm $\|\varphi\|_{\mathcal{R}_2^{(\nu)}} := \|\varphi^{(\nu)}\|_2$.

Variants of the space $\mathcal{R}_2^{(\nu)}$, like $\mathcal{D}^{-\nu}(L_2([0,1])$ for instance, have been considered as appropriate models to work out problems in fractional Brownian motion, see [FP], [Hu] (note that also the space $\mathcal{T}_2^{(\nu)}(t^\nu) = W^\nu(L_2(\mathbb{R}^+))$ lies in that setting, [Hu, p. 5]). The covariance or kernel associated with the space $\mathcal{R}_2^{(\nu)}$ is given by the very well known formula

$$\mathfrak{n}_\nu(t,s) = \int_0^{t\wedge s} \frac{(t-u)^{\nu-1}}{\Gamma(\nu)} \frac{(s-u)^{\nu-1}}{\Gamma(\nu)} du,$$

with $\nu > 0$ referring to the number of "times" that we integrate the white noise (see [FP], [SL]). ($\mathfrak{b}_\nu = \mathfrak{n}_{\nu+1}$).

Also we have

$$\mathfrak{n}_\nu(t,s) = \frac{(ts)^{\nu-1}(t\wedge s)}{\Gamma(\nu)\Gamma(\nu+1)} {}_2F_1\left(1-\nu, 1, \nu+1, \frac{t\wedge s}{t\vee s}\right)$$

whence

$$\mathfrak{n}_\nu(t,s) = (t\vee s)(t\wedge s)(ts)^{\nu-1} k_\nu(t,s) = (ts)^\nu k_\nu(t,s).$$

We next show a natural isometry between $\mathcal{T}_2^{(\nu)}(t^\nu)$ and $\mathcal{R}_2^{(\nu)}$. This isometry is perhaps part of the folklore, but we have been unable to find a place where to get it explicitly.

Lemma 4.2.1. *Let* $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$ *and define* $\varphi(x) = x^{\nu-1} f\left(\frac{1}{x}\right)$ *for* $x > 0$. *Then*

$$\varphi \in \mathcal{R}_2^{(\nu)} \quad \text{and} \quad \varphi^{(\nu)}(x) = x^{-(\nu+1)} W^\nu f\left(\frac{1}{x}\right), \quad x > 0.$$

Proof. For f, φ as in the statement,

$$\begin{aligned} \varphi(x) = x^{\nu-1} f\left(\frac{1}{x}\right) \Leftrightarrow \varphi(x) &= \frac{x^{\nu-1}}{\Gamma(\nu)} \int_{1/x}^\infty \left(t - \frac{1}{x}\right)^{\nu-1} W^\nu f(t) dt \\ &= \frac{1}{\Gamma(\nu)} \int_{1/x}^\infty \left(x - \frac{1}{t}\right)^{\nu-1} t^{\nu-1} W^\nu f(t) dt \\ &= \frac{1}{\Gamma(\nu)} \int_0^x (x-y)^{\nu-1} y^{-(\nu+1)} W^\nu f\left(\frac{1}{y}\right) dy, \end{aligned}$$

with

$$\int_0^\infty \left| y^{-(\nu+1)} W^\nu f\left(\frac{1}{y}\right)\right|^2 dy = \int_0^\infty |t^\nu W^\nu f(t)|^2 dt < \infty.$$

Thus the proof is over. □

Remark 4.2.2.

$$\begin{array}{ccc} L^2(0,\infty) & \longrightarrow & L^2(0,\infty) \\ f(x) & \mapsto & \frac{1}{x} f\left(\frac{1}{x}\right) \end{array}$$

is an isometric isomorphism.

Here is the isometry.

Proposition 4.2.3. *The mapping defined by*

$$\begin{array}{cccc} \Theta_\nu : & \mathcal{T}_2^{(\nu)}(t^\nu) & \longrightarrow & \mathcal{R}_2^{(\nu)} \\ & f & \mapsto & x^{\nu-1} f\left(\frac{1}{x}\right) \end{array}$$

is an isometric isomorphism.

Proof. By Lemma 4.2.1, Θ_ν is well defined; moreover, it is obviously injective. As for the surjectivity, note that if $\varphi \in \mathcal{R}_2^{(\nu)}$ then $\varphi^{(\nu)} \in L_2(\mathbb{R}^+)$, that is, $\frac{1}{x}\varphi^{(\nu)}\left(\frac{1}{x}\right) \in L_2(\mathbb{R}^+)$, by Remark 4.2.2. Therefore, there exists a unique $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$ such that $\frac{1}{x}\varphi^{(\nu)}\left(\frac{1}{x}\right) = x^\nu W^\nu f(x)$ and we know from Lemma 4.2.1 that

$$\varphi^{(\nu)}(x) = x^{-(\nu+1)} W^\nu f\left(\frac{1}{x}\right) \Leftrightarrow \varphi(x) = x^{\nu-1} f\left(\frac{1}{x}\right),$$

as we wanted to show. □

Corollary 4.2.4.

$$\mathcal{R}_2^{(\nu)} \hookrightarrow L^2(\mathbb{R}^+, x^{-2\nu})$$

Proof. For $\varphi \in \mathcal{R}_2^{(\nu)}$,

$$\int_0^\infty |\varphi(x)|^2 \frac{dx}{x^{2\nu}} = \int_0^\infty \left| x^{\nu-1} f\left(\frac{1}{x}\right)\right|^2 \frac{dx}{x^{2\nu}} = \int_0^\infty \left|\frac{1}{x} f\left(\frac{1}{x}\right)\right|^2 dx = \int_0^\infty |f(x)|^2 dx < \infty$$

for a (unique) $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$. Moreover,

$$\begin{aligned} \|\varphi\|_{L_2(x^{-2\nu})} &= \|f\|_2 \le M_\nu \|f\|_{2,(\nu)} = M_\nu \int_0^\infty |t^\nu W^\nu f(t)|^2 dt \\ &= M_\nu \int_0^\infty \left| x^{-(\nu+1)} W^\nu f\left(\frac{1}{x}\right)\right|^2 dx = M_\nu \|\varphi^{(\nu)}\|_2 = M_\nu \|\varphi\|^2_{\mathcal{R}_2^{(\nu)}}. \end{aligned}$$

□

Remark 4.2.5. By Proposition 1.3.2 (3), $\mathcal{T}_2^{(\mu)}(t^\mu) \hookrightarrow \mathcal{T}_2^{(\nu)}(t^\nu)$ for $\mu > \nu$. However, there is no continuous embeddings between the spaces $\mathcal{R}_2^{(\mu)}, \mathcal{R}_2^{(\nu)}$. We see it with an example. Let $F(t) = t/(1+t)$. Then $F \notin L_2(\mathbb{R}^+)$, but $F'(t) = 1/(1+t)^2 \in L_2(\mathbb{R}^+)$, and $F \in \mathcal{R}_2^{(1)}$.

If we transform $f(s) = F\left(\frac{1}{s}\right) = \frac{1}{1+s}$, we have $f \in L_2(\mathbb{R}^+), sf'(s) = \frac{-s}{(1+s)^2} \in L_2(\mathbb{R}^+)$, therefore $f \in \mathcal{T}_2^{(1)}(t)$.

Remark 4.2.6. Note that, for $\nu > 1/2$ and $\varphi \in \mathcal{R}_2^{(\nu)}$, with $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$ such that $\varphi(x) = x^{\nu-1} f\left(\frac{1}{x}\right)$, we have $\lim_{\varepsilon\downarrow 0} \varphi(\varepsilon) = \lim_{\varepsilon\downarrow 0} \varepsilon^{\nu-1} f(1/\varepsilon) = \lim_{t\to\infty} f(t) t^{1-\nu} = 0$ by (1.13).

The space $\mathcal{R}_2^{(1)}$ can be found for example in [PR, p. 149] in the form

$$\mathcal{R}_2^{(1)} = \{f : [0, +\infty) \to \mathbb{C} \mid f \text{ absolutely continuous, } f(0) = 0, f' \in L^2\},$$

or in [SS, p. 14] under the description

$$\mathcal{R}_2^{(1)} = \{f \in W^{1,2}(0,\infty) \mid \lim_{\varepsilon\downarrow 0} f(\varepsilon) = 0\},$$

where $W^{1,2}(0,\infty)$ is the Sobolev space of differential order 1 based on L^2. According to [BT, p. 243], $\mathbb{R}_2^{(1)}$ is called the Cameron-Martin space and its unit ball known as the Strassen set. For $\nu = n \in \mathbb{N}$, one has that $\mathcal{R}_2^{(n)}$ is isometrically isomorphic to the subspace of the Sobolev space $W^{n,2}(0,1)$ which is formed by the functions f in $W^{n,2}(0,1)$ satisfying the boundary conditions $f^{(j)}(0) = 0$, $j = 0, \cdots, n-1$, see [BT, p. 92].

The Laplace transform of functions in $\mathcal{R}_2^{(\nu)}$ is easy to obtain. For $r \in \mathbb{R}$, put $\zeta_r(z) := z^r = e^{r \log z}$, $z \in \mathbb{C}^+$, where $\log z$ is the principal branch of the logarithm with principal argument in $[-\pi, \pi)$.

Corollary 4.2.7. *For all $\varphi \in \mathcal{R}_2^{(\nu)}$ and $z \in \mathbb{C}^+$,*

$$\mathcal{L}(\varphi^{(\nu)})(z) = z^\nu \mathcal{L}(\varphi)(z).$$

Moreover

$$\mathcal{L}(\mathcal{R}_2^{(\nu)}) = \zeta_{-\nu} H_2(\mathbb{C}^+).$$

Proof. For $\varphi \in \mathcal{R}_2^{(\nu)}$ we have $\varphi = (\mathfrak{r}_\nu * \varphi^{(\nu)})$. Hence,

$$\mathcal{L}(\varphi) = \mathcal{L}\left(\mathfrak{r}_\nu * \varphi^{(\nu)}\right) = \mathcal{L}\left(\mathfrak{r}_\nu\right)\mathcal{L}(\varphi^{(\nu)}) = \zeta_{-\nu}\mathcal{L}(\varphi^{(\nu)}),$$

which proves the first equality. Thus we have $\mathcal{L}(\mathcal{R}_2^{(\nu)}) \subseteq \zeta_{-\nu}H_2(\mathbb{C}^+)$.

Conversely, suppose that F is a holomorphic function in $\mathbb{C}^+$ such that $\zeta_{-\nu}F \in H_2(\mathbb{C}^+)$. Then, by Paley-Wiener's theorem [Ru, Th.19.2], there is $\phi \in L_2(\mathbb{R}^+)$ such that $\mathcal{L}(\phi) = \zeta_\nu F$. Hence, $F = \zeta_{-\nu}\mathcal{L}(\phi) = \mathcal{L}(\mathfrak{r}_\nu)\mathcal{L}(\phi) = \mathcal{L}(\mathfrak{r}_\nu * \phi) = \mathcal{L}(\varphi)$ with $\varphi = \mathfrak{r}_\nu * \phi \in \mathcal{R}_2^{(\nu)}$ and the proof is over. □

Corollary 4.2.7 is the Paley-Wiener type theorem which corresponds to the Hilbert space $\mathcal{L}(\mathcal{R}_2^{(\nu)})$. It is not so simple to find the Laplace transform of elements in $\mathcal{T}_2^{(\nu)}(t^\nu)$. We deal with this question in the next section.

4.3 Hardy-Sobolev spaces

Recall, for $1 \le p < \infty$ and $F \in H_p(\mathbb{C}^+)$,

$$T_p(t)F(z) := e^{-t/p}F(e^{-t}z), \quad t \in \mathbb{R}, z \in \mathbb{C}^+.$$

is a C_0-group of isometries on $H_p(\mathbb{C}^+)$.

Let $\mathfrak{C}_\nu^*$ be the Cesàro-Hardy operator, introduced in Definition 1.2.3, given by

$$\mathfrak{C}_\nu^* F := \int_0^\infty \varphi_p(t)T_p(-t)F\,dt \ \in H_p(\mathbb{C}^+),$$

where $\varphi_p(t) := (1-e^{-t})^{\nu-1}e^{-t/p}$; $\nu > 0$, $t > 0$. As seen after that definition, $\mathfrak{C}_\nu^* F(z) = \mathcal{C}_\nu^* F_\theta(|z|)$ for $F \in H_2(\mathbb{C}^+)$, $z \in \mathbb{C}^+$, whence it follows that $\mathfrak{C}_\nu^*$ is injective.

Definition 4.3.1. *For $1 \le p < \infty$, define the Hardy-Sobolev space, of order $\nu > 0$, $H_p^{(\nu)}(\mathbb{C}^+)$ by*

$$H_p^{(\nu)}(\mathbb{C}^+) := \mathfrak{C}_\nu^*(H_p(\mathbb{C}^+)),$$

endowed with the norm $\|F\|_{p,(\nu)} := \|(\mathfrak{C}_\nu^*)^{-1}F\|_p$, $F \in H_p^{(\nu)}(\mathbb{C}^+)$.

Then, in analogy to the real case, put

$$\mathfrak{W}^\nu := \frac{1}{\Gamma(\nu+1)}[\zeta_{-\nu} \circ (\mathfrak{C}_\nu^*)^{-1}]$$

or, equivalently,

$$\mathfrak{W}^{-\nu} = \frac{1}{\Gamma(\nu+1)}\mathfrak{C}_\nu^* \circ \zeta_\nu,$$

where ζ_ν is the mutiplication operator by z^ν, for $\nu \in \mathbb{R}$.

With this operational notation, we have that $F \in H_p^{(\nu)}(\mathbb{C}^+)$ if and only if there exists $\mathfrak{W}^\nu F$ holomorphic in $\mathbb{C}^+$, with $\zeta_\nu \mathfrak{W}^\nu F \in H_p(\mathbb{C}^+)$, such that

$$F(z) = \frac{1}{\Gamma(\nu)} \int_z^\infty (\lambda - z)^{-1} \mathfrak{W}^\nu F(\lambda) d\lambda, \ z \in \mathbb{C}^+, \tag{4.5}$$

where the integration path is the ray joinng z with the complex infinity point. Also, $\|F\|_{p,(\nu)} := \|\zeta_\nu W^\nu F\|_p$.

Indeed, working on rays leaving the origin in $\overline{\mathbb{C}^+}$, one gets, like in (1.12),

$$\mathfrak{W}^\eta F(z) = \frac{1}{\Gamma(\nu - \eta)} \int_z^\infty (\lambda - z)^{\nu-\eta-1} \mathfrak{W}^\nu F(\lambda) d\lambda, \ z \in \mathbb{C}^+,$$

for every η such that $0 \leq \eta < \nu$. From here, one obtains the continuous inclusions

$$H_p^{(\mu)}(\mathbb{C}^+) \hookrightarrow H_p^{(\nu)}(\mathbb{C}^+) \hookrightarrow H_p^{(0)}(\mathbb{C}^+) = H_p(\mathbb{C}^+)$$

for all $\mu > \nu$. In particular we have $\|F\|_{p,(\nu)} \leq C_{p,\nu} \|F\|_p$, $F \in H_p^{(\nu)}(\mathbb{C}^+)$.

From now on in this chapter, we consider the case $p = 2$. We know that $H_2(\mathbb{C}^+)$ is a RKHS with kernel $K(z,w) = (z + \overline{w})^{-1}$; $z, w \in \mathbb{C}^+$. Since $\|F\|_{2,(\nu)} \leq C_\nu \|F\|_2$ for every $F \in H_2^{(\nu)}(\mathbb{C}^+)$ one has that point evaluations

$$\mathrm{ev}_z \colon H_2^{(\nu)}(\mathbb{C}^+) \hookrightarrow H_2(\mathbb{C}^+), \quad z \in \mathbb{C}^+,$$

are continuous on $H_2^{(\nu)}(\mathbb{C}^+)$ and so this space is a RKHS. Let K_ν denote its reproducing kernel, so that $K_{\nu,w}$ belongs to $H_2^{(\nu)}(\mathbb{C}^+)$, where

$$K_{\nu,w}(z) := K_\nu(z,w), \quad z, w \in \mathbb{C}^+.$$

The aim of this section is to establish the following theorem. Its first part provides an integral expression for the kernel K_ν. The second part is a Paley-Wiener type result.

Theorem 4.3.2. *Let $\nu > 0$.*

(1) *The space $H_2^{(\nu)}(\mathbb{C}^+)$ is a RKHS with reproducing kernel*

$$K_\nu(z,w) = \int_0^1 \int_0^1 \frac{(1-x)^{\nu-1}}{\Gamma(\nu)} \frac{(1-y)^{\nu-1}}{\Gamma(\nu)} \frac{1}{xz + y\overline{w}} dx dy, \ z, w \in \mathbb{C}^+.$$

(2) *The Laplace transform is an isometric isomorphism from $\mathcal{T}_2^{(\nu)}(t^\nu)$ onto $H_2^{(\nu)}(\mathbb{C}^+)$.*

We prove the theorem using the basis method.

Let $(\ell_m)_{m \geq 0}$ be the orthonormal basis of Laguerre polynomials in $L_2(\mathbb{R}^+)$ and let $\ell_{m,\nu} := W^{-\nu}(t^{-\nu}\ell_m)$ be the orthonormal basis in $\mathcal{T}_2^{(\nu)}(t^\nu)$ obtained from $(\ell_m)_{m\geq 0}$ via the isometry $W^{-\nu}(t^{-\nu}(\cdot))$, both given in the beginning of Section 4.1.

Lemma 4.3.3. *For $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}^+$,*

(1) $$(\mathcal{L}\ell_m)(z) = \frac{2(2z-1)^m}{(2z+1)^{m+1}}.$$

(2) $$\mathcal{L}(\ell_{m,\nu})(z) = \frac{2}{\Gamma(\nu)} \int_1^\infty \frac{(u-1)^{\nu-1}}{u^\nu} \frac{(2z-u)^m}{(2z+u)^{m+1}} du.$$

Proof. (1) For $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}^+$,

$$\begin{aligned}
(\mathcal{L}\ell_m)(z) &= \int_0^\infty e^{-zu} e^{-u/2} \sum_{j=0}^m \binom{m}{j} (-1)^j \frac{u^j}{j!} du \\
&= \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{1}{j!} \frac{j!}{(z+\frac{1}{2})^{j+1}} = \sum_{j=0}^m \binom{m}{j} \left(-\frac{2}{2z+1}\right)^{j+1} \\
&= \frac{2}{2z+1}\left(1-\frac{2}{2z+1}\right)^m = \frac{2(2z-1)^m}{(2z+1)^{m+1}}.
\end{aligned}$$

(2) Also,

$$\begin{aligned}
\mathcal{L}(\ell_{m,\nu})(z) &= \sum_{j=0}^m \binom{m}{j} \frac{(-1)^j}{j!} \int_1^\infty \int_0^\infty t^j e^{-t(\frac{u}{2}+z)} dt \frac{(u-1)^{\nu-1}}{u^{\nu-j}} \frac{du}{\Gamma(\nu)} \\
&= \frac{1}{\Gamma(\nu)} \int_1^\infty \frac{(u-1)^{\nu-1}}{u^\nu} \sum_{j=0}^m \binom{m}{j} \left(\frac{-u}{\frac{u}{2}+z}\right)^j \frac{du}{\frac{u}{2}+z} \\
&= \frac{1}{\Gamma(\nu)} \int_1^\infty \frac{2(u-1)^{\nu-1}}{u^\nu(u+2z)} \left(1-\frac{2u}{u+2z}\right)^m du \\
&= \frac{2}{\Gamma(\nu)} \int_1^\infty \frac{(u-1)^{\nu-1}}{u^\nu} \frac{(2z-u)^m}{(2z+u)^{m+1}} du.
\end{aligned}$$

□

Since $(\mathcal{L}(\ell_m))_{m\geq 0}$ is an orthonormal basis in $H_2(\mathbb{C}^+)$ and the mapping

$$H_2(\mathbb{C}^+) \to H_2^{(\nu)}(\mathbb{C}^+),\ F \mapsto \mathfrak{W}^{-\nu}(\zeta^{-\nu} F)$$

is an isometry, it follows that

$$\mathfrak{L}_{m,\nu} := \mathfrak{W}^{-\nu}(\zeta^{-\nu}\mathcal{L}\ell_m),\ m = 0, 1, \ldots,$$

form an orthonormal basis in $H_2^{(\nu)}(\mathbb{C}^+)$.

Lemma 4.3.4. *For $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}^+$,*

$$\mathfrak{L}_{m,\nu}(z) = \frac{2}{\Gamma(\nu)} \int_1^\infty \frac{(u-1)^{\nu-1}}{u^\nu} \frac{(2uz-1)^m}{(2uz+1)^{m+1}} du = \frac{(-1)^m}{4z} \mathcal{L}(\ell_{m,\nu}) \left(\frac{1}{4z}\right).$$

Proof. Let $z = |z|e^{i\theta} \in \mathbb{C}^+$, $m \in \mathbb{N} \cup \{0\}$. Then, integrating on the ray (path) given by θ and applying Lemma 4.3.3 (1),

$$\begin{aligned}
\mathfrak{L}_{m,\nu}(z) &= \frac{2}{\Gamma(\nu)} \int_z^\infty \frac{(\lambda - z)^{\nu-1}}{\lambda^\nu} \frac{(2\lambda - 1)^m}{(2\lambda + 1)^{m+1}} d\lambda \\
&= \frac{2}{\Gamma(\nu)} \int_z^\infty z^{\nu-1} \frac{(\lambda z^{-1} - 1)^{\nu-1}}{\lambda^\nu} \frac{(2\lambda - 1)^m}{(2\lambda + 1)^{m+1}} d\lambda \\
&\overset{u=\lambda z^{-1}}{=} \frac{2}{\Gamma(\nu)} \int_1^\infty \frac{(u - 1)^{\nu-1}}{u^\nu} \frac{(2uz - 1)^m}{(2uz + 1)^{m+1}} du \\
&= \frac{2(-1)^m}{4z\Gamma(\nu)} \int_1^\infty \frac{(u - 1)^{\nu-1}}{u^\nu} \frac{(2(1/4z) - u)^m}{(2(1/4z) + u)^{m+1}} du \\
&= \frac{(-1)^m}{4z} \mathcal{L}(\ell_{m,\nu}) \left(\frac{1}{4z} \right).
\end{aligned}$$

□

Let us now consider the space $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$. Since $\mathcal{L}$ is a one-to-one mapping we define the norm $\|F\|_{\mathcal{L}} := \|f\|_{2,(\alpha)}$ on $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$, where $F = \mathcal{L}(f)$, $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$. Clearly, $(\mathcal{L}(\ell_{m,\nu}))_{m,\nu}$ is an orthonormal basis in $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$. Further, by Proposition 1.3.2 (3) and the classical Paley-Wiener theorem -that is, $H_2(\mathbb{C}^+) = \mathcal{L}(L_2(\mathbb{R}^+))$- one has

$$\mathcal{L}(\mathcal{T}_2^{(\mu)}(t^\mu)) \hookrightarrow \mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu)) \hookrightarrow H_2(\mathbb{C}^+), \ \forall \mu \geq \nu \geq 0.$$

Thus in particular we know that $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$ is a RKHS.

Proposition 4.3.5. *For $\nu > 0$, the reproducing kernel Q_ν of $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$ is represented by the integral*

$$Q_{\nu,w}(z) := Q_\nu(z, w) = \int_0^\infty G_\nu(z, r)\overline{G_n u(w, r)} dr, \quad z, w \in \mathbb{C}^+$$

where

$$G_\nu(z, r) = \frac{1}{\Gamma(\nu + 1)} \mathcal{C}_\nu(e_z)(r) = \frac{1}{r^\nu} \int_0^r \frac{(r - u)^{\nu-1}}{\Gamma(\nu)} e^{-zu} du, \quad z \in \mathbb{C}^+, r > 0.$$

Proof. Let $w \in \mathbb{C}^+$. By definition, there exists a unique $h_{\nu,w} \in \mathcal{T}_2^{(\nu)}(t^\nu)$ such that $Q_{\nu,w} = \mathcal{L}(h_{\nu,w})$ in $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$. Take $f \in \mathcal{T}_2^{(\nu)}(t^\nu)$ and $F = \mathcal{L}f$ in $\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))$. Then by Hardy's inequality we have that the integral $\int_0^\infty s^\nu |W^\nu f(s)| \left(\frac{1}{s^\nu} \int_0^s (s - t)^{\nu-1} e^{-(\mathrm{Re}\, w)t} dt \right) ds$ is finite. Hence one can apply Fubini's theorem (in the last-but-one equality of the following chain) to obtain

$$\begin{aligned}
(\tau^\nu W^\nu f | \tau^\nu W^\nu h_{\nu,w})_2 &= (f | h_{\nu,w})_{2,(\nu)} = (F | Q_{\nu,w})_{\mathcal{L}(\mathcal{T}_2^{(\nu)}(t^\nu))} = F(w) = \mathcal{L}f(w) \\
&= \int_0^\infty \int_t^\infty W^\nu f(s) \frac{(s - t)^{\nu-1}}{\Gamma(\nu)} ds \ e^{-wt} dt \\
&= \int_0^\infty s^\nu W^\nu f(s) \left(\frac{1}{s^\nu} \int_0^s \frac{(s - t)^{\nu-1}}{\Gamma(\nu)} e^{-wt} dt \right) ds.
\end{aligned}$$

Hence $Q_{\nu,w} = \mathcal{L}(h_{\nu,w})$ with $h_{\nu,w}$ such that

$$s^\nu W^\nu h_{\nu,w}(s) = \frac{1}{\Gamma(\nu+1)}\mathcal{C}_\nu(e_{\overline{w}})(s), \ s > 0;$$

that is,

$$h_{\nu,w}(u) = \int_r^\infty \frac{(u-r)^{\nu-1}}{\Gamma(\nu+1)}\mathcal{C}_\nu(e_{\overline{w}})(u)\frac{dr}{\Gamma(\nu)} = \frac{1}{\Gamma(\nu+1)^2}\mathcal{C}_\nu^*(\mathcal{C}_\nu e_{\overline{w}})(u), \ u > 0.$$

Therefore

$$\begin{aligned} Q_\nu(z,w) &= \frac{\mathcal{L}(h_{\nu,w})(z)}{\Gamma(\nu+1)^2} = \frac{(\mathcal{C}_\nu^*\mathcal{C}_\nu e_{\overline{w}}|e_{\overline{z}})_2}{\Gamma(\nu+1)^2} = \frac{(\mathcal{C}_\nu e_{\overline{w}}|\mathcal{C}_\nu e_{\overline{z}})_2}{\Gamma(\nu+1)^2} \\ &= \int_0^\infty \left(\int_0^r \frac{(r-s)^{\nu-1}}{r^\nu}e^{-zs}\frac{ds}{\Gamma(\nu)}\right)\left(\int_0^r \frac{(r-t)^{\nu-1}}{r^\nu}e^{-\overline{w}t}\frac{dt}{\Gamma(\nu)}\right)dr, \end{aligned}$$

for every $z, w \in \mathbb{C}^+$, as we wanted to show. $\square$

Remark 4.3.6. For $\nu > 1/2$, the above proposition is an immediate consequence of [S1, p. 82-83] when applied to $\mathcal{L}\colon \mathcal{T}_2^{(\nu)}(t^\nu) \to H_2^{(\nu)}(\mathbb{C}^+)$ since $\mathcal{T}_2^{(\nu)}(t^\nu)$ is RKHS with kernel $\int_0^\infty g_\nu(s,r)\overline{g_\nu(t,r)}dr$, $s, t > 0$, and $G_\nu = \mathcal{L}(g_\nu)$. Recall, however, that $\mathcal{T}_2^{(\nu)}(t^\nu)$ is not a RKHS for $0 < \nu \le 1/2$, so one needs an argument as the above one in the proposition, in this case.

Proof of Theorem 4.3.2. Let $z, w \in \mathbb{C}^+$. By Proposition 4.3.5,

$$\begin{aligned} Q_\nu(z,w) &= \frac{1}{\Gamma(\nu+1)^2}\int_0^\infty \mathcal{C}_\nu(e_z)(r)\mathcal{C}_\nu(e_w)(r)dr \\ &= \int_0^\infty \left(\int_0^r \frac{(r-s)^{\nu-1}}{\Gamma(\nu)}\frac{e^{-zs}}{r^\nu}ds\right)\left(\int_0^r \frac{(r-t)^{\nu-1}}{\Gamma(\nu)}\frac{e^{-\overline{w}t}}{r^\nu}dt\right)dr \\ &= \int_0^\infty \int_0^1 \frac{(1-x)^{\nu-1}}{\Gamma(\nu)}\int_0^1 \frac{(1-y)^{\nu-1}}{\Gamma(\nu)}e^{-zrx}e^{-\overline{w}ry}\ dx\ dy\ dr \\ &= \int_0^1 \frac{(1-x)^{\nu-1}}{\Gamma(\nu)}\int_0^1 \frac{(1-y)^{\nu-1}}{\Gamma(\nu)}\int_0^\infty e^{-s(xz+y\overline{w})}\ ds\ dy\ dx \\ &= \int_0^1\int_0^1 \frac{(1-x)^{\nu-1}}{\Gamma(\nu)}\frac{(1-y)^{\nu-1}}{\Gamma(\nu)}\frac{1}{xz+y\overline{w}}\ dy\ dx. \end{aligned}$$

(We have applied Fubini's theorem; this argument is granted because the latter integral is clearly finite with $\operatorname{Re} z, \operatorname{Re} w$ instead of $z, \overline{w}$, which implies the finiteness of the triple integral of the module of the functions involved.)

On the other hand,

$$
\begin{aligned}
K_\nu(z,w) &= \sum_{j=0}^{\infty} \mathfrak{L}_{j,\nu}(z)\overline{\mathfrak{L}_{j,\nu}(w)} = \frac{1}{4z\overline{w}} \sum_{j=0}^{\infty} \mathcal{L}(\ell_{j,\nu})\left(\frac{1}{4z}\right)\overline{\mathcal{L}(\ell_{j,\nu})\left(\frac{1}{4w}\right)} \\
&= \frac{1}{4z\overline{w}} \int_0^1\int_0^1 \frac{(1-x)^{\nu-1}(1-y)^{\nu-1}}{\Gamma(\nu)^2} \frac{dx\,dy}{(x/4z)+(y/4\overline{w})} \\
&= \int_0^1\int_0^1 \frac{(1-x)^{\nu-1}(1-y)^{\nu-1}}{\Gamma(\nu)^2} \frac{dx\,dy}{x\overline{w}+yz} = Q_\nu(z,w),
\end{aligned}
$$

where the first and third equality hold true by [S1, Th. 1.8] and the second one follows from Lemma 4.3.4.

Since the reproducing kernels K_ν and Q_ν are equal the Hilbert spaces that they generate coincide, and the proof is over. □

Remark 4.3.7. Theorem 4.3.2 (2) extends Theorem 3.3 of [GMMS] to fractional ν. The proofs are rather different. In [GMMS], the identity $\mathcal{L}(\mathcal{T}_2^{(n)}(t^n)) = H_2^{(n)}(\mathbb{C}^+)$, $n \in \mathbb{N}$, relies on the usage of Laguerre polynomials, as we will see in Appendix 4.6.

4.4 Estimating the kernel

We next proceed to estimate the norm of the kernel function K_ν. For $\nu \geq 1$, the calculation of such an estimate is the same as that one done for $\nu = n \in \mathbb{N}$ in [GMMS]. We include it here for the sake of completeness.

To begin with, one needs to know the value of the following standard integral.

Lemma 4.4.1. *For $\theta \in (-\pi/2, \pi/2)$,*

$$
J(\theta) := \int_0^1 \frac{1}{t^2+1+2t\cos 2\theta}\,dt = \frac{|\theta|}{|\sin 2\theta|}, \text{ if } \theta \neq 0; \quad J(0) = 1/2.
$$

Proof. For $\theta = 0$, one gets $J(0) = 1/2$ very easily. For $0 < |\theta| < \frac{\pi}{2}$, we have

$$
\begin{aligned}
J(\theta) &= \int_0^1 \frac{dt}{(t+\cos(2\theta))^2+\sin^2(2\theta)} \\
&= \frac{1}{|\sin(2\theta)|} \int_0^{1/|\sin(2\theta)|} \left[\left(r + \frac{\cos(2\theta)}{|\sin(2\theta)|}\right)^2 + 1\right]^{-1} dr \\
&= \frac{1}{|\sin(2\theta)|} \int_{\cos(2\theta)/|\sin(2\theta)|}^{\cos\theta/|\sin\theta|} \frac{du}{u^2+1} = \frac{|\theta|}{|\sin(2\theta)|},
\end{aligned}
$$

□

Theorem 4.4.2. *Let $\nu > 0$. Then, for every $z = |z|e^{i\theta} \in \mathbb{C}^+$,*

(1) *For* $\nu \geq 1$,

$$\frac{1}{(2\nu-1)\Gamma(\nu)^2}\ \frac{1}{|z|} \leq \|K_\nu(\cdot,z)\|^2_{2,(\nu)} \leq \frac{\pi}{\nu\Gamma(\nu)^2}\ \frac{1}{|z|}.$$

(2) *For* $1/2 < \nu < 1$,

$$\frac{1}{\Gamma(\nu)^2}\ \frac{1}{|z|} \leq \|K_\nu(\cdot,z)\|^2_{2,(\nu)} \leq \frac{\pi}{(2\nu-1)\Gamma(\nu)^2}\ \frac{1}{|z|}.$$

(3) *For* $0 < \nu \leq 1/2$,

$$\frac{1}{\Gamma(\nu)^2}\frac{1}{|z|} \leq \|K_\nu(\cdot,z)\|^2_{2,(\nu)} \leq \frac{2}{\Gamma(\nu+1)^2}\frac{1}{\operatorname{Re} z},\ \ \textit{if } |\theta| \leq \pi/4,$$

and

$$\frac{1}{\Gamma(\nu)^2}\frac{1}{|z|} \leq \|K_\nu(\cdot,z)\|^2_{2,(\nu)} \leq \frac{2}{\Gamma(\nu+1)^2}\frac{1}{\operatorname{Re} z},\ \ \textit{if } \pi/4 < |\theta| < \pi/2.$$

Proof. Let $z = |z|e^{i\theta} \in \mathbb{C}^+$ with $\theta \in (-\pi/2, \pi/2)$. For every $\nu > 0$ one has

$$\begin{aligned}\|K_{\nu,z}\|^2_{2,(\nu)} &= K_\nu(z,z) = \int_0^1\int_0^1 \frac{(1-y)^{\nu-1}}{\Gamma(\nu)}\frac{(1-x)^{\nu-1}}{\Gamma(\nu)}\frac{1}{x\overline{z}+yz}dxdy\\ &= \frac{1}{|z|}\int_0^1\int_0^1 \frac{(1-y)^{\nu-1}}{\Gamma(\nu)}\frac{(1-x)^{\nu-1}}{\Gamma(\nu)}\frac{(x+y)\cos\theta}{x^2+y^2+2xy\cos(2\theta)}dxdy,\end{aligned}$$

since $\operatorname{Im}(K_{\nu,z}(z)) = 0$ (use symmetry in the imaginary part of the integral). Thus using symmetry again (with respect to the diagonal in $(0,1)\times(0,1)$) and then the change of variables $x = yt$ one obtains

$$\begin{aligned}K_\nu(z,z) &= \frac{2\cos\theta}{\Gamma(\nu)^2|z|}\int_0^1\int_0^y \frac{(1-x)^{\nu-1}(1-y)^{\nu-1}(x+y)}{x^2+y^2+2xy\cos(2\theta)}dxdy\\ &\overset{(*)}{=} \frac{2\cos\theta}{\Gamma(\nu)^2|z|}\int_0^1\left(\int_0^1(1-yt)^{\nu-1}(1-y)^{\nu-1}dy\right)\frac{1+t}{t^2+1+2t\cos(2\theta)}dt.\end{aligned}$$

(1) If $\nu \geq 1$ then

$$\int_0^1 (1-yt)^{\nu-1}(1-y)^{\nu-1}dy \leq \int_0^1 (1-y)^{\nu-1}dy = \frac{1}{\nu}$$

and therefore, by equality $(*)$,

$$K_\nu(z,z) \leq \frac{4\cos\theta}{\nu\Gamma(\nu)^2|z|}J(\theta) = \frac{2}{\nu\Gamma(\nu)^2}\frac{|\theta|}{|\sin\theta|}\frac{1}{|z|} \leq \frac{\pi}{\nu\Gamma(\nu)^2}\frac{1}{|z|}.$$

For the lower estimate, one has $(1-yt)^{\nu-1} \geq (1-y)^{\nu-1}$ for $0 < t, y < 1$ and so

$$\begin{aligned}K_\nu(z,z) &\geq \frac{2\cos\theta}{\Gamma(\nu)^2|z|}\int_0^1\int_0^1 \frac{(1-y)^{2\nu-2}}{t^2+1+2t\cos(2\theta)}dtdy\\ &= \frac{2}{\Gamma(\nu)^2(2\nu-1)}\frac{|\theta|}{|\sin\theta|}\frac{1}{|z|} \geq \frac{1}{\Gamma(\nu)^2(2\nu-1)}\frac{1}{|z|}.\end{aligned}$$

(2) If $1/2 < \nu < 1$, using the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}\int_0^1 (1-yt)^{\nu-1}(1-y)^{\nu-1}dy &\leq \frac{1}{2\nu-1}\left(\frac{1-(1-t)^{2\nu-1}}{t}\right)^{1/2} \\ &\leq \left(\sup_{0<t<1}\frac{1-(1-t)^{2\nu-1}}{(2\nu-1)t}\right) = \frac{1}{2\nu-1}.\end{aligned}$$

and then, by $(*)$,

$$K_\nu(z,z) \leq \frac{4\cos\theta}{\Gamma(\nu)^2(2\nu-1)}J(\theta)\frac{1}{|z|} \leq \frac{\pi}{\Gamma(\nu)^2(2\nu-1)}\frac{1}{|z|}.$$

As for the lower estimate, from $(*)$, one has

$$\begin{aligned}K_\nu(z,z) &\geq \frac{2\cos\theta}{\Gamma(\nu)^2|z|}\int_0^1\int_0^1 \frac{dy\ dt}{t^2+1+2t\cos(2\theta)} \\ &= \frac{2\cos\theta}{\Gamma(\nu)^2|z|}J(\theta) \geq \frac{1}{\Gamma(\nu)^2}\frac{1}{|z|}.\end{aligned}$$

(3) Assume $0 < \nu \leq 1/2$. We now deal first with the lower estimate. Then, as in (2),

$$K_\nu(z,z) \geq \frac{2\cos\theta}{\Gamma(\nu)^2|z|}J(\theta) = \frac{1}{\Gamma(\nu)^2}\frac{1}{|z|}.$$

For the upper estimate, by $(*)$,

$$\begin{aligned}K_\nu(z,z) &\leq \frac{4\cos\theta}{\Gamma(\nu)^2|z|}\left(\int_0^1 (1-y)^{\nu-1}dy\right)\left(\int_0^1 \frac{(1-t)^{\nu-1}}{t^2+1+2t\cos(2\theta)}dt\right) \\ &= \frac{4\cos\theta}{\nu\Gamma(\nu)^2|z|}\int_0^1 \frac{(1-t)^{\nu-1}}{t^2+1+2t\cos(2\theta)}dt\end{aligned}$$

Now is the moment to notice that $\cos(2\theta) \geq 0$ if $|\theta| \leq \pi/4$ whereas $\cos(2\theta) < 0$ if $|\theta| > \pi/4$.

For $|\theta| \leq \pi/4$,

$$K_\nu(z,z) \leq \frac{4\cos\theta}{\nu\Gamma(\nu)^2|z|}\int_0^1 \frac{(1-t)^{\nu-1}}{t^2+1}dt \leq \frac{2\cos\theta}{\Gamma(\nu+1)^2|z|} \leq \frac{2}{\Gamma(\nu+1)^2|z|}$$

For $|\theta| > \pi/4$,

$$\begin{aligned}K_\nu(z,z) &\leq \frac{4\cos\theta}{\nu\Gamma(\nu)^2|z|}\int_0^1 \frac{(1-t)^{\nu-1}}{(t+\cos(2\theta))^2+\sin^2(2\theta)}dt \leq \frac{4\cos\theta}{\nu\Gamma(\nu)^2|z|}\int_0^1 \frac{(1-t)^{\nu-1}}{\sin^2(2\theta)}dt \\ &= \frac{1(1/\sin^2\theta)}{\Gamma(\nu+1)^2\cos\theta}\frac{1}{|z|} \leq \frac{2}{\Gamma(\nu+1)^2}\frac{1}{\operatorname{Re} z}.\end{aligned}$$

The proof is over. $\square$

In the above theorem, parts (1) and (2), we have shown that

$$\|K_\nu(\cdot,z)\|_{2,(\nu)} \simeq \frac{1}{\sqrt{|z|}},\ z\in\mathbb{C}^+,$$

up to constants only depending on ν, for $\nu>1/2$.

QUESTION: In Theorem 4.4.2 (3), so for $0<\nu\le 1/2$,

$$\text{Is } \|K_\nu(\cdot,z)\|_{2,(\nu)} \simeq \frac{1}{\sqrt{|z|}},\ z\in\mathbb{C}^+,$$

or

$$\text{is } \|K_\nu(\cdot,z)\|_{2,(\nu)} \simeq \frac{1}{\sqrt{\operatorname{Re} z}},\ z\in\mathbb{C}^+?$$

Or maybe none of the above holds true ?

Remark 4.4.3. Recall from Corollary 4.2.7 that $\mathcal{L}(\mathcal{R}_2^{(\nu)})=\zeta_{-\nu}H_2(\mathbb{C}^+)$. Endowed with the norm $\|\zeta_\nu F\|_2$, $F\in\zeta_{-\nu}H_2(\mathbb{C}^+)$, the space $\mathcal{L}(\mathcal{R}_2^{(\nu)})$ is a RKHS. It is readily seen that the reproducing kernel of $\mathcal{L}(\mathcal{R}_2^{(\nu)})$ is given by $(z,\omega)\in\mathbb{C}^+\times\mathbb{C}^+\mapsto (z\overline{\omega})^{-\nu}(z+\overline{\omega})^{-1}\in\mathbb{C}$.

4.5 Averaging Brownian motion

One can also introduce a Hilbert space of absolutely continuous functions of fractional order, using the Cesàro-Hardy operator $\mathcal{C}_\nu$ in a similar manner to which we have done using $\mathcal{C}_\nu^*$ above. We focus on the case $p=2$ as usual in this chapter.

Set $\mathcal{T}^2_{(\nu)}:=\mathcal{C}_\nu(L_2(\mathbb{R}^+))$. Since $\mathcal{C}_\nu$ is injective by Tichsmarsh's theorem we are allowed to define the norm $\|f\|_{(\nu),2}:=\|(\mathcal{C}_\nu)^{-1}f\|_2$, $\forall f\in\mathcal{T}^2_{(\nu)}$.

Thus $\mathcal{T}^2_{(\nu)}$ is a Hilbert space with inner product

$$(f|g)_{(\nu),2}=\int_0^\infty ((\mathcal{C}_\nu)^{-1}f)(s)\overline{((\mathcal{C}_\nu)^{-1}g)(s)}ds,\ f,g\in\mathcal{T}^2_{(\nu)},$$

and such that $\mathcal{T}^2_{(\nu)}\hookrightarrow L_2(\mathbb{R}^+)$. Moreover, the formula

$$f(t)=\frac{\nu}{t^\nu}\int_0^t (t-s)^{\nu-1}((\mathcal{C}_\nu)^{-1}f)(s)ds$$

entails that $\mathcal{T}^2_{(\nu)}$ is a RKHS for $\nu>1/2$. Note that, in the notation of Section 4.2, $(\tau_\nu f)^{(\nu)}=\Gamma(\nu+1)\mathcal{C}_\nu^{-1}(f)$.

Set $H^2_{(\nu)}(\mathbb{C}^+):=\mathcal{L}(\mathcal{T}^2_{(\nu)})$ endowed with the inner product

$$(\mathcal{L}f|\mathcal{L}g)_{(\nu),2}:=(f|g)_{(\nu),2}=(\mathcal{C}_\nu^{-1}f|\mathcal{C}_\nu^{-1}g)_2.$$

Since $H^2_{(\nu)}(\mathbb{C}^+) \hookrightarrow H_2(\mathbb{C}^+)$ the space $H^2_{(\nu)}(\mathbb{C}^+)$ is a RKHS for every $\nu > 0$. Let N_ν its reproducing kernel. There is a unique $\phi_{w,\nu} \in \mathcal{T}^2_{(\nu)}$ such that $\mathcal{L}(\phi_{w,\nu}) = N_{\nu,w}$. Then, for $w \in \mathbb{C}^+$ and every $f \in \mathcal{T}^2_{(\nu)}$,

$$\begin{aligned}(\mathcal{C}_\nu^{-1} f|\mathcal{C}_\nu^{-1} g)_2 &= (\mathcal{L}f|N_{\nu,w})_{(\nu),2} = \mathcal{L}f(w) \\ &= (\mathcal{C}_\nu\mathcal{C}_\nu^{-1} f)|e_{\overline{w}})_2 = (\mathcal{C}_\nu^{-1} f|\mathcal{C}_\nu^*(e_{\overline{w}}))_2,\end{aligned}$$

whence we derive $\mathcal{C}_\nu^{-1}\phi_{w,\nu} = \mathcal{C}_\nu^*(e_{\overline{w}})$ from which

$$\phi_{w,\nu} = \mathcal{C}_\nu\mathcal{C}_\nu^*(e_{\overline{w}}) = \mathcal{C}_\nu^*\mathcal{C}_\nu(e_{\overline{w}}).$$

by Corollary 1.2.2.

In other words, we have got $\phi_{w,\nu} = \Gamma(\nu+1)^2 h_{\nu,w}$ where $h_{\nu,w}$ is the function obtained prior to Remark 4.3.6. This implies that $N_\nu(z,w) = \Gamma(\nu+1)^2 Q_\nu(z,w) = \Gamma(\nu+1)^2 K_\nu(z,w)$ for $z, w \in \mathbb{C}^+$ which is to say that the spaces $H^2_{(\nu)}(\mathbb{C}^+)$ and $H_2^{(\nu)}(\mathbb{C}^+)$ are the same (and so are $\mathcal{T}_2^{(\nu)}(t^\nu)$ and $\mathcal{T}^2_{(\nu)}$) ! In particular, we have got the equivalence of the norms $\|(\tau_\nu f)^\nu\|_2$ and $\|\tau_\nu W^\nu f\|_2$ for $f \in \mathcal{T}^2_{(\nu)}$, and therefore we have extended [GMMS, Prop. 2.6] to fractional derivatives.

Remark 4.5.1. Note that the argument followed prior to this remark allows us to identify the spaces $\mathcal{L}(\mathcal{C}_\nu(L_2(\mathbb{R}^+)))$ and $\mathcal{L}(\mathcal{T}_\nu^{(\nu)}(t^\nu))$ –previous identification of $\phi_{w,\nu}$ and $\Gamma(\nu+1)^2 h_{\nu,w}$– *independently* of Theorem 4.3.2. In fact, part (2) of such a theorem is easily obtained from the equality $\mathcal{L}(\mathcal{C}_\nu(L_2(\mathbb{R}^+))) = \mathcal{L}(\mathcal{T}_\nu^{(\nu)}(t^\nu))$ using Corollary 1.2.6:

$$\begin{aligned}\mathcal{L}(\mathcal{C}_\nu(L_2(\mathbb{R}^+))) &= (\mathcal{L}\circ\mathcal{C}_\nu)(L_2(\mathbb{R}^+)) = (\mathfrak{C}_\nu\circ\mathcal{L})(L_2(\mathbb{R}^+)) \\ &= \mathfrak{C}_\nu(\mathcal{L}(L_2(\mathbb{R}^+))) = \mathfrak{C}_\nu(H_2(\mathbb{C}^+)) = H_2^{(\nu)}(\mathbb{C}^+)\end{aligned}$$

where we have used that corollary in the second equality and the classical Paley-Wiener theorem in the last-but-one equality.

We have however chosen to present our first approach to Theorem 4.3.2 to give a more complete description of the space $H_2^{(\nu)}(\mathbb{C}^+)$, by showing a nice basis in it. Recall in passing that having on hand suitable bases in RKHS of the fBm's may be helpful to get representations of Gaussian processes, see [Hu, p.3].

On the other hand, self-similarity is an important topic related with fBm's (and where in particular Hardy spaces on $\mathbb{C}^+$ of fractional derivatives of Laplace transforms are of interest, see [Mb, p. 277]. In this respect, notice that the kernels (covariances) k_ν and K_ν of preceding sections satisfy, for $\lambda > 0$, $s, t > 0$ and $z, w \in \mathbb{C}^+$, the rules

$$k_\nu(\lambda s, \lambda t) = \lambda^{-1} k_\nu(s,t); \qquad K_\nu(\lambda z, \lambda w) = \lambda^{-1} K_\nu(z,w)$$

independently of $\nu > 0$.

In conclusion, on account of the properties of spaces $\mathcal{T}_2^{(\nu)}(t^\nu)$ pointed out in previous sections and the simple but interesting description of the space $H^{(\nu)}(\mathbb{C}^+)$ of Laplace transforms of the elements in $\mathcal{T}_2^{(\nu)}(t^\nu)$, we wonder if operating with averages of fractal Brownian processes $t^{-\nu}\int_0^t (t-s)^{\nu-1} B_s ds$ or $t^{-\nu}\int_0^t (t-s)^{\nu-1} h(s) ds$, $h \in L_2(\mathbb{R}^+)$, could be helpful in this setting, at least from an operational viewpoint.

4.6 Appendix: Hardy-Sobolev spaces of integer order

The Paley-Wiener type result given in Theorem 4.3.2 (2) is an extension to fractional order $\nu > 0$ of the analogue theorem for $\nu = n \in \mathbb{N}$ which had been stablished in [GMMS, Th.6.2]. I think that the argument used in [GMMS] is of sufficient interest to show it on this Appendix.

In the next lemma, for functions $f \in \mathcal{T}_2^{(n)}(t^n)$, $t^n f^{(n)}(t)$ is expressed as a combination of $((t^k f)^k)_{0\le k\le n}$ and, vice versa, $(t^n f)^n$ is written as a combination of $((t^k f)^k)_{0\le k\le n}$. This property allows us to give another equivalent norm to $\| \ \|_{2,(n)}$. The proof is omitted because it is just the aplication of Leibniz rule for the n-th derivative of a product.

Lemma 4.6.1. *For $n \geq 0$, we define the coefficients $c_{i,j}$, $0 \le i, j \le n$, by*

$$c_{i,j} = \begin{cases} 0, & \text{for } i < j; \\ \binom{i}{j}\dfrac{i!}{j!}, & \text{for } i \geq j. \end{cases}$$

Then, for $f \in \mathcal{T}_2^{(n)}(t^n)$ and $t > 0$, we have that

$$\begin{aligned} (t^n f)^{(n)}(t) &= \sum_{k=0}^{n} c_{n,k} t^k f^{(k)}(t), \\ t^n f^{(n)}(t) &= \sum_{k=0}^{n} (-1)^{k+n} c_{n,k} (t^k f)^{(k)}(t). \end{aligned}$$

Remarks 4.6.2. The following properties of the coefficients are included just in the pursuit of completeness, but they will not be used.

(1) If we considered the $(n+1)$-square matrix $C_n = (c_{i,j})_{0\le i,j\le n}$, it is easy to prove that C_n is invertible, and $C_n^{-1} = ((-1)^{i+j} c_{i,j})_{0\le i,j\le n}$.

(2) If we define the sequence $c_n = \sum_{k=0}^{n} c_{n,k}$ for $n \geq 0$. Then the first values of the sequence $(c_n)_{n\geq 0}$ are 1, 2, 7, 34, 209, 1546.... This sequence appears in "The On-Line Encyclopedia of Integer Sequences" by N.J.A. Sloane with the reference A002720.

Lemma 4.6.3. *Let n be a positive integer number.*

(1) *We have $(e_\lambda)_{\mathrm{Re}\,\lambda>0} \subset \mathcal{T}_2^{(n)}(t^n)$ and*

$$\|e_\lambda\|_{2,(n)} = \left(\frac{(2n)!}{2^{2n+1}}\right)^{\frac{1}{2}} \frac{|\lambda|^n}{(\mathrm{Re}\,\lambda)^{n+\frac{1}{2}}}, \qquad \lambda \in \mathbb{C}^+.$$

(2) *The set span$\{e_\lambda \ : \ \mathrm{Re}\,\lambda > 0\}$ is dense in $\mathcal{T}_2^{(n)}(t^n)$.*

Proof. The proof of part (1) is straightforward. Now take $f \in \mathcal{T}_2^{(n)}(t^n)$ such that $\langle f, e_\lambda\rangle_{\mathcal{T}_2^{(n)}(t^n)} = 0$ for $\lambda \in \mathbb{C}^+$, i.e.,

$$0 = (-1)^n\lambda^n \int_0^\infty t^{2n}e^{-\lambda t}f^{(n)}(t)dt = \lambda^n\mathcal{L}(t^n f^{(n)})^{(n)}(\lambda), \quad \lambda \in \mathbb{C}^+.$$

Since the Laplace transform $\mathcal{L}$ is one-to-one, we conclude that $t^n f^{(n)}(t) = 0$ in $L_2(\mathbb{R}^+)$, $f = 0$ in $\mathcal{T}_2^{(n)}(t^n)$ and the set $\mathrm{span}\{e_\lambda \ : \ \mathrm{Re}\,\lambda > 0\}$ is dense in $\mathcal{T}_2^{(n)}(t^n)$. □

In the next proposition we obtain an equivalent expression for the inner product given in (4.1). Here Laguerre polynomials and Legendre polynomials play a central role.

Proposition 4.6.4. *Let $f, g \in \mathcal{T}_2^{(n)}(t^n)$. Then*

$$(f|g)_{2,(n)} = \left((t^n f)^{(n)}\middle|(t^n g)^{(n)}\right)_2. \tag{4.6}$$

Proof. Let $e_\lambda(t) := e^{-\lambda t}$ and $e_\mu(t) := e^{-\mu t}$, with $\lambda, \mu > 0$. Then

$$\left(t^n e_\lambda^{(n)}\middle|t^n e_\mu^{(n)}\right)_2 = (\lambda\mu)^n \int_0^\infty t^{2n}e^{-(\lambda+\mu)t}dt = \frac{(\lambda\mu)^n(2n)!}{(\lambda+\mu)^{2n+1}}.$$

For the second inner product in (4.6), consider that

$$(t^n e_\lambda)^{(n)}(t) = n!\mathrm{L}_n(\lambda t)$$

where L_n is the Laguerre polynomial of degree n. Therefore

$$\left((t^n e_\lambda)^{(n)}\middle|(t^n e_\mu)^{(n)}\right)_2 = (n!)^2\int_0^\infty \mathrm{L}_n(\lambda t)\mathrm{L}_n(\mu t)e^{-(\lambda+\mu)t}dt.$$

Now then, according to [GR, 7.414 (2)],

$$\int_0^\infty \mathrm{L}_n(\lambda x)\mathrm{L}_n(\mu x)e^{-bx}dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}}\mathrm{P}_n\left(\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right),$$

for $\mathrm{Re}\,b > 0$, where P_n is the Legendre polynomial of degree n. Therefore

$$\begin{aligned}\left(t^n e_\lambda)^{(n)}\middle|(t^n e_\mu)^{(n)}\right)_2 &= (n!)^2\lim_{b\to\lambda+\mu}\frac{(b-\lambda-\mu)^n}{b^{n+1}}\mathrm{P}_n\left(\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right)\\ &= (n!)^2\lim_{b\to\lambda+\mu}\frac{(b-\lambda-\mu)^n}{b^{n+1}}\frac{(2n)!}{2^n(n!)^2}\left(\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right)^n = \frac{(2n)!(\lambda\mu)^n}{(\lambda+\mu)^{2n+1}}\end{aligned}$$

because $\frac{(2n)!}{2^n(n!)^2}$ is the leading coefficient of P_n, as can be seen in [MOS, Section 5.4.2]. We conclude that

$$(e_\lambda|e_\mu)_{2,(n)} = \left((t^n e_\lambda)^{(n)}\middle|(t^n e_\mu)^{(n)}\right)_2, \quad \lambda, \mu \in \mathbb{C}^+$$

and by linearity $(s|r)_{2,(n)} = \left((t^n s)^{(n)} \Big\| (t^n r)^{(n)}\right)_2$, for $s, r \in \operatorname{span}\{e_\lambda : \operatorname{Re}\lambda > 0\}$. Finally take $f, g \in \mathcal{T}_2^{(n)}(t^n)$ and $(r_j)_{j\geq 0}, (s_k)_{j\geq 0} \subset \operatorname{span}\{e_\lambda : \operatorname{Re}\lambda > 0\}$ such that $r_j \to f$ and $s_k \to g$ in $\mathcal{T}_2^{(n)}(t^n)$ (Lemma 4.6.3 (2)). Since

$$(t^n f)^{(n)} = \sum_{k=0}^{n} \binom{n}{k}^2 (n-k)! t^k f^{(k)}, \qquad t > 0,$$

and then

$$\|(t^n f)^{(n)}\|_2 \leq C \|t^n f^{(n)}\|_2,$$

we have that $(t^n r_j)^{(n)} \to (t^n f)^{(n)}$ and $(t^n s_k)^{(n)} \to (t^n g)^{(n)}$ in $L_2(\mathbb{R}^+)$ and

$$(f|g)_{2,(n)} \leftarrow (r_j|s_k)_{2,(n)} = \left((t^n r_j)^{(n)} \Big| (t^n s_k)^{(n)}\right)_2 \to \left((t^n f)^{(n)} \Big| (t^n g)^{(n)}\right)_2,$$

and we conclude the result. □

We show now another proof of the Paley-Wiener theorem in this setting. For this, we need a couple of simple observations.

Recall the following properties of the Laplace transform:

$$\mathcal{L}(t^n h) = (-1)^n (\mathcal{L}h)^{(n)}, \quad t^n h \in L_2(\mathbb{R}^+), \tag{4.7}$$

and

$$z^n \mathcal{L}h(z) = \mathcal{L}(h^{(n)})(z) + \sum_{j=0}^{n-1} z^{n-1-j} h^{(j)}(0), \qquad z \in \mathbb{C}^+ \tag{4.8}$$

with $h, h^{(n)} \in L_2(\mathbb{R}^+)$ and $h, h^{(1)}, \ldots, h^{(n-1)}$ continuous in $[0, \infty)$. We combine these identities and Lemma 4.6.1 to obtain the following result, which is given, for Sobolev algebras, in [GMR1, Lemma 4.1].

Lemma 4.6.5. *For every $n \in \mathbb{N}$, $z \in \overline{\mathbb{C}^+} \setminus \{0\}$ and $f \in \mathcal{T}_2^{(n)}$,*

$$\begin{aligned} z^k (\mathcal{L}f)^{(k)}(z) &= \sum_{j=0}^{k} (-1)^k \binom{k}{j} \frac{k!}{j!} \mathcal{L}(t^j f^{(j)})(z), \qquad k = 0, 1, \ldots, n; \\ \mathcal{L}(t^k f^{(k)})(z) &= \sum_{j=0}^{k} (-1)^k \binom{k}{j} \frac{k!}{j!} z^j (\mathcal{L}(f))^{(j)}(z), \qquad k = 0, 1, \ldots, n; \end{aligned}$$

Theorem 4.6.6. *Let $n \in \mathbb{N}$. The Laplace transform $\mathcal{L} : \mathcal{T}_2^{(n)}(t^n) \to H_2^{(n)}(\mathbb{C}^+)$ is an isometric isomorphism between Hilbert spaces, i.e., $F, G \in H_2^{(n)}(\mathbb{C}^+)$ if and only if there exist unique $f, g \in \mathcal{T}_2^{(n)}(t^n)$ such that $F = \mathcal{L}f$ and $G = \mathcal{L}g$ such that*

$$(f|g)_{2,(n)} = (\mathcal{L}f|\mathcal{L}g)_{2,(n)}, \qquad f, g \in \mathcal{T}_2^{(n)}(t^n).$$

Proof. Given $f \in \mathcal{T}_2^{(n)}(t^n)$, then $t^k f^{(k)} \in L^2(\mathbb{R}^+)$ and $z^k(\mathcal{L}f)^{(k)} \in H_2(\mathbb{C}^+)$ (by Lemma 4.6.5) for $0 \le k \le n$, i.e., $\mathcal{L}f \in H_2^{(n)}(\mathbb{C}^+)$.

Now, take $F \in H_2^{(n)}(\mathbb{C}^+)$. By the usual Paley-Wiener theorem, there exists $f \in L_2(\mathbb{R}^+)$ such that $F = \mathcal{L}f$. We aim is to conclude that, in fact, $f \in \mathcal{T}_2^{(n)}(t^n)$. We denote by G the analytic function defined by

$$G(z) := \sum_{j=0}^{n} (-1)^n \binom{n}{j} \frac{n!}{j!} z^j F^{(j)}(z), \qquad z \in \mathbb{C}^+.$$

Since $F \in H_2^{(n)}(\mathbb{C}^+)$, we get that $G \in H_2(\mathbb{C}^+)$ and there exists $g \in L_2(\mathbb{R}^+)$ such that $G = \mathcal{L}g$. It is clear that $t^{-n}g \in L^2(t^{2n})$ and the function $\tilde{f} \in \mathcal{T}_2^{(n)}(t^n)$ where

$$\tilde{f}(s) := \frac{1}{(n-1)!} \int_s^\infty \frac{(s-t)^{n-1}}{t^n} g(t)dt, \qquad s > 0.$$

By (1.10), note that $g = t^n(\tilde{f})^{(n)}$. Now we apply Lemma 4.6.5 to get that

$$G(z) = \mathcal{L}g(z) = \mathcal{L}(t^n(\tilde{f})^{(n)})(z) = \sum_{j=0}^{n} (-1)^n \binom{n}{j} \frac{n!}{j!} z^j \mathcal{L}(\tilde{f})^{(j)}(z), \qquad z \in \mathbb{C}^+.$$

Since $F = \mathcal{L}f$, we conclude that

$$\sum_{j=0}^{n} (-1)^n \binom{n}{j} \frac{n!}{j!} z^j \mathcal{L}(\tilde{f} - f)^{(j)}(z) = 0, \qquad z \in \mathbb{C}^+.$$

By Lemma 4.6.1, $\left(z^n \mathcal{L}(\tilde{f} - f)\right)^{(n)}(z) = \sum_{j=0}^{n} \binom{n}{j} \frac{n!}{j!} z^j \mathcal{L}(\tilde{f} - f)^{(j)}(z) = 0$ for $z \in \mathbb{C}^+$, and we conclude that $z^n \mathcal{L}(\tilde{f} - f)(z) = P_n(z)$, where P_n is a polynomial of order equal or less n. Then $\mathcal{L}(\tilde{f} - f) \in H_2$ and bounded, we conclude that $\mathcal{L}(\tilde{f} - f) = 0$, and $f = \tilde{f} \in \mathcal{T}_2^{(n)}(t^n)$.

Let $F, G \in H_2^{(n)}(\mathbb{C}^+)$. By the first part of the proof, there exist $f, g \in \mathcal{T}_2^{(n)}(t^n)$ such that $F = \mathcal{L}f$ and $G = \mathcal{L}g$. We apply Proposition 4.6.4 and the classical Paley-Wiener Theorem to get

$$
\begin{aligned}
(f|g)_{2,(n)} &= \left(t^n f^{(n)}\middle|t^n g^{(n)}\right)_2 = \left((t^n f)^{(n)}\middle|(t^n g)^{(n)}\right)_2 \\
&= \left(\mathcal{L}((t^n f)^{(n)})\middle|\mathcal{L}((t^n g)^{(n)})\right)_2 \\
&= \Bigg(\mathcal{L}((t^n f)^{(n)})(z) + \sum_{j=0}^{n-1} z^{n-1-j}(t^n f)^{(j)}(0)\Bigg| \\
&\qquad\qquad \mathcal{L}((t^n g)^{(n)})(z) + \sum_{j=0}^{n-1} z^{n-1-j}(t^n g)^{(j)}(0)\Bigg)_2 \\
&= \left(z^n\mathcal{L}(t^n f)(z)\middle|z^n\mathcal{L}(t^n g)(z)\right)_2 = \left(z^n(\mathcal{L}f)^{(n)}(z)\middle|z^n(\mathcal{L}g)^{(n)}(z)\right)_2 \\
&= (F|G)_{2,(n)}
\end{aligned}
$$

where we have used that $\lim_{x\to 0^+} x^{k+1}f^{(k)}(x) = 0$, for all $0 \leq k \leq n-1$ and $f \in \mathcal{T}_2^{(n)}(t^n)$. □

A consequence of Theorem 4.6.6 is an analogous result to Proposition 4.6.4 in the space $H_2^{(n)}(\mathbb{C}^+)$ for $n \geq 1$.

Corollary 4.6.7. *Let $(H_2^{(n)}(\mathbb{C}^+), (\cdot|\cdot)_{2,(n)})$ the Hardy-Sobolev space introduced in Definition 4.3.1. Then*

$$
(F|G)_{2,(n)} = ((z^n F)^{(n)}|(z^n G)^{(n)})_2, \qquad F, G \in H_2^{(n)}(\mathbb{C}^+).
$$

Proof. Take $F, G \in H_2^{(n)}(\mathbb{C}^+)$. By Theorem 4.6.6, there exist $f, g \in \mathcal{T}_2^{(n)}(t^n)$ such that $\mathcal{L}f = F$ and $\mathcal{L}g = G$. Moreover we apply Proposition 4.6.4 to get

$$
\begin{aligned}
(F|G)_{2,(n)} &= (f|g)_{2,(n)} = \left(t^n f^{(n)}\middle|t^n g^{(n)}\right)_2 = (\mathcal{L}(t^n f^{(n)})|\mathcal{L}(t^n g^{(n)}))_2 \\
&= ((\mathcal{L}f^{(n)})^{(n)}|(\mathcal{L}(g^{(n)})^{(n)}))_2 = ((z^n F)^{(n)}|(z^n G)^{(n)})_2
\end{aligned}
$$

where we have applied equalities (4.7) and (4.8). □

4.6.1 Composition operators C_φ

Here we study the behaviour of composition operators on the spaces $H_2^{(n)}(\mathbb{C}^+)$. Our approach is limited to the particular case $n \in \mathbb{N}$, since we do not have suitable tools to address the problem for general $\nu > 0$. However, even in the integer case the characterization of the boundedness of composition operators (which is the wished goal) does not seem to be simple. We give some partial results.

From the bounds for the kernel norms given in Theorem 4.4.2 we obtain the following

Proposition 4.6.8. *Let $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ analytic such that the composition operator $C_\varphi : H_2^{(n)}(\mathbb{C}^+) \to H_2^{(n)}(\mathbb{C}^+)$, with $C_\varphi f = f \circ \varphi$, is a bounded operator. Then*

$$\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty.$$

Proof. If C_φ is a bounded operator, so it is the adjoint C_φ^*, and there exists $M > 0$ such that $\|C_\varphi^*\| \le M$. Take $z \in \mathbb{C}^+$. Then

$$\begin{aligned} \frac{1}{\Gamma(n)^2(2n-1)|\varphi(z)|} &\le \|K_n(\cdot,\varphi(z))\|_{2,(n)}^2 = \|C_\varphi^* K_n(\cdot,z)\|_{2,(n)}^2 \\ &\le M^2\|K_n(\cdot,z)\|_{2,(n)}^2 \le \frac{M^2\pi}{n\Gamma(n)^2|z|} \end{aligned}$$

and finally

$$\frac{|z|}{|\varphi(z)|} \le \frac{M^2\pi(2n-1)}{n}.$$

Note that we have used that

$$\begin{aligned} \big(f|C_\varphi^* K_n(\cdot,w)\big)_{2,(n)} &= (C_\varphi f|K_n(\cdot,w))_{2,(n)} = C_\varphi f(\omega) = f\circ\varphi(\omega) \\ &= f(\varphi(\omega)) = (f|K_n(\cdot,\varphi(\omega)))_{2,(n)} \end{aligned}$$

and therefore $C_\varphi^* K_n(\cdot,w) = K_n(\cdot,\varphi(\omega))$. □

This proposition immediately suggests the following implication between composition operators on $H_2^{(n)}(\mathbb{C}^+)$ and on $H_2(\mathbb{C}^+)$:

Corollary 4.6.9. *Let $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ analytic such that induces a bounded composition operator C_φ on $H_2^{(n)}(\mathbb{C}^+)$. Then φ induces a composition operator C_φ on $H_2(\mathbb{C}^+)$.*

Proof. From the previous corollary,

$$\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty.$$

Now we apply a Julia-Caratheodory theorem in $\mathbb{C}^+$ and Theorem 3.1 from [EJ] to get that C_φ is bounded on $H_2(\mathbb{C}^+)$. □

Lemma 4.6.10. *Let $n \in \mathbb{N}$, $p \ge 1$, $\varphi : U \to V$ and $f : W \to \mathbb{C}$, with $\varphi \in \mathcal{H}(U)$, $f \in \mathcal{H}(W)$, $U, V, W \subseteq \mathbb{C}$, $V \subseteq W$, U and W open sets.*

$$\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty.$$

Then

$$|z|^{np}\left|\frac{d^n}{dz^n}(f\circ\varphi)(z)\right|^p \leq C_{n,p}\sum\Bigg(|f^{(m_1+\cdots+m_n)}(\varphi(z))|^p|\varphi(z)|^{(m_1+\cdots+m_n)p} \cdot\prod_{j=1}^n|\varphi^{(j)}(z)|^{m_jp}|\varphi(z)|^{(j-1)m_jp}\Bigg)$$

for all $z\in U$, where the sum is over all n-tuples of nonnegative integers $(m_1,\cdots,m_n)$ satisfying the constraint

$$m_1+2m_2+\cdots+nm_n=n. \tag{4.9}$$

Proof. The main tool in the proof is the well-known Faà di Bruno's formula for the n-th derivative of a composition of functions.

$$\frac{d^n}{dz^n}f(\varphi(z))=\sum\left(\frac{n!}{m_1!\cdots m_n!}f^{(m_1+\cdots+m_n)}(\varphi(z))\prod_{j=1}^n\left(\frac{\varphi^{(j)}(z)}{j!}\right)^{m_j}\right)$$

where the sum is over all n-tuples of nonnegative integers $(m_1,\cdots,m_n)$ satisfying (4.9). Let $n\in\mathbb{N}$, $p\geq 1$ and $z\in U$. Assume $C_0>0$ such that

$$\frac{|z|}{|\varphi(z)|}\leq C_0,\qquad\text{for all } z\in U,$$

$$\begin{aligned}
&|z|^{np}\left|\frac{d^n}{dz^n}f(\varphi(z))\right|^p\\
&= \frac{|z|^{np}}{|\varphi(z)|^{np}}\left|\sum\left(\frac{n!}{m_1!\cdots m_n!}f^{(m_1+\cdots+m_n)}(\varphi(z))\prod_{j=1}^n\left(\frac{\varphi^{(j)}(z)}{j!}\right)^{m_j}\right)\right|^p|\varphi(z)|^{np}\\
&\leq C_0^{np}\left|\sum\left(\frac{n!}{m_1!\cdots m_n!}f^{(m_1+\cdots+m_n)}(\varphi(z))\prod_{j=1}^n\left(\frac{\varphi^{(j)}(z)}{j!}\right)^{m_j}\right)\right|^p|\varphi(z)|^{np}\\
&\leq C_0^{np}\left(\sum\left(\frac{n!}{m_1!\cdots m_n!}|f^{(m_1+\cdots+m_n)}(\varphi(z))|\prod_{j=1}^n\left(\frac{|\varphi^{(j)}(z)|}{j!}\right)^{m_j}\right)\right)^p|\varphi(z)|^{np}\\
&\leq C_{n,p}\sum\left(\frac{n!}{m_1!\cdots m_n!}|f^{(m_1+\cdots+m_n)}(\varphi(z))|\prod_{j=1}^n\left(\frac{|\varphi^{(j)}(z)|}{j!}\right)^{m_j}\right)^p|\varphi(z)|^{np}\\
&\leq C_{n,p}\sum\left(|f^{(m_1+\cdots+m_n)}(\varphi(z))|^p\prod_{j=1}^n|\varphi^{(j)}(z)|^{m_jp}\right)|\varphi(z)|^{np}\\
&= C_{n,p}\sum\Bigg(|f^{(m_1+\cdots+m_n)}(\varphi(z))|^p|\varphi(z)|^{(m_1+\cdots+m_n)p}\\
&\qquad\cdot\prod_{j=1}^n|\varphi^{(j)}(z)|^{m_jp}|\varphi(z)|^{(j-1)m_jp}\Bigg),
\end{aligned}$$

where we have applied the particular case of the Jensen's inequality (for $a_i \geq 0$ and $p \geq 1$)

$$\left(\sum_{i=1}^{N} a_i\right)^p \leq N^{p-1} \sum_{i=1}^{N} a_i^p$$

and the fact that (4.9) implies

$$np = (m_1 + \cdots + m_n)p + \sum_{j=1}^{n} (j-1)m_j p.$$

□

Recall that a positive measure ν on $\mathbb{C}^+$ is called a *Carleson measure* if there is a constant $N_\nu > 0$ such that

$$\nu(Q_{t,h}) \leq N_\nu h$$

for all squares

$$Q_{t,h} := \{(x,y) \in \mathbb{C}^+ \ : \ 0 < x < h, \ t < y < t+h\}, \quad t \in \mathbb{R}, \ h > 0,$$

see for example [G, p.30].

Theorem 4.6.11. *Let $n \in \mathbb{N}$ and $p \geq 1$. Let $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function such that*

$$\sup_{z \in \mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty.$$

Assume that the family $\{\nu_{\varphi,M,x}\}_{x>0}$ is a set of Carleson measures with common bound, where

$$\nu_{\varphi,M,x}(E) = \int_{\{y \in \mathbb{R} \ : \ \varphi(x+iy) \in E\}} \Phi_{\varphi,M}(x+iy)dy,$$

with $\Phi_{\varphi,M}(x+iy) = \prod_{j=1}^{n} |\varphi^{(j)}(x+iy)|^{m_j p} |\varphi(x+iy)|^{(j-1)m_j p}$ for $E \in \mathcal{B}or(\mathbb{C}^+)$ and each $M = (m_1, \cdots, m_n)$ satisfying (4.9).

Then φ induces a bounded composition operator C_φ on $H_p^{(n)}(\mathbb{C}^+)$.

Proof. Let $f \in H_p^{(n)}(\mathbb{C}^+)$. Then $C_\varphi f = f \circ \varphi \in \mathcal{H}(\mathbb{C}^+)$. By Lemma 4.6.10,

$$\begin{aligned}
\|C_\varphi f\|_{p,(n)}^p &= \sup_{x>0} \int_{-\infty}^{\infty} \Big|\frac{d^n}{dz^n}(f\circ\varphi)(x+iy)\Big|^p |x+iy|^{np} dy \\
&\le C_{n,p} \sup_{x>0} \int_{-\infty}^{\infty} \sum \Big(|f^{(m_1+\dots+m_n)}(\varphi(x+iy))|^p |\varphi(x+iy)|^{(m_1+\dots+m_n)p} \\
&\qquad \cdot \prod_{j=1}^n |\varphi^{(j)}(x+iy)|^{m_j p} |\varphi(x+iy)|^{(j-1)m_j p} \Big) dy \\
&\le C_{n,p} \sup_{x>0} \sum \Big(\int_{-\infty}^{\infty} |f^{(m_1+\dots+m_n)}(\varphi(x+iy))|^p |\varphi(x+iy)|^{(m_1+\dots+m_n)p} \\
&\qquad \cdot \prod_{j=1}^n |\varphi^{(j)}(x+iy)|^{m_j p} |\varphi(x+iy)|^{(j-1)m_j p} dy \Big).
\end{aligned}$$

Now for each one of the integrals we use a well-known measure theory theorem in order to change the measure (see for example [H], p.163) and we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} |f^{(m_1+\dots+m_n)}(\varphi(x+iy))|^p |\varphi(x+iy)|^{(m_1+\dots+m_n)p} \\
&\qquad \cdot \prod_{j=1}^n |\varphi^{(j)}(x+iy)|^{m_j p} |\varphi(x+iy)|^{(j-1)m_j p} dy \\
&= \iint_{\mathbb{C}^+} |\omega|^{(m_1+\dots+m_n)p} |f^{(m_1+\dots+m_n)}(\omega)| d\nu_{\varphi,p,M_k,x}(\omega).
\end{aligned}$$

The fact that the family $\{\nu_{\varphi,M,x}\}_{x>0}$ is a set of Carleson measures implies (see [G], p.61)

$$\begin{aligned}
&\iint_{\mathbb{C}^+} |f^{(m_1+\dots+m_n)}(\omega)| |\omega|^{(m_1+\dots+m_n)p} d\nu_{\varphi,p,M_k,x}(\omega) \\
&\qquad \le C_\varphi \|\omega^{m_1+\dots+m_n} f^{(m_1+\dots+m_n)}(\omega)\|_p^p = C_\varphi \|f\|_{p,(m_1+\dots+m_n)}^p.
\end{aligned}$$

Finally we use that $m_1 + \dots + m_n \le n$ and the embedding between the spaces $H_p^{(n)}(\mathbb{C}^+)$ to get

$$\|C_\varphi f\|_{p,(n)}^p \le C_{\varphi,n,p} \|f\|_{p,(n)}^p.$$

□

Note that we have the result just for a fixed p, not for each $1 \le p < \infty$. We can give sufficient (and more tractable) conditions to get the boundedness for all p.

Corollary 4.6.12. *Let $n \in \mathbb{N}$ and let $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function such that*

$$\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty \quad \textit{and} \quad \sup_{z\in\mathbb{C}^+, 1\le j\le n} |\varphi^{(j)}(z)\varphi^{j-1}(z)| < \infty.$$

Assume that the family $\{\nu_{\varphi,x}\}_{x>0}$ *is a set of Carleson measures with common bound, where*

$$\nu_{\varphi,x}(E) = |\{y \in \mathbb{R} \ : \ \varphi(x+iy) \in E\}|$$

for $E \in \mathcal{B}or(\mathbb{C}^+)$ *and* $|\cdot|$ *denotes the Lebesgue measure on* $\mathbb{R}$.

Then φ *induces a bounded composition operator* C_φ *on* $H_p^{(n)}(\mathbb{C}^+)$*, for each* $1 \leq p < \infty$.

Proof. Similarly to the proof of Theorem 4.6.11, and by using (2), we get

$$\|C_\varphi f\|_{p,(n)}^p \leq C_{n,p} \sup_{x>0} \sum \left(\int_{-\infty}^{\infty} |f^{(m_1+\cdots+m_n)}(\varphi(x+iy))|^p |\varphi(x+iy)|^{(m_1+\cdots+m_n)p} dy \right),$$

and we end in a similar way by applying in each integral the change of measure, the fact that $\{\nu_{\varphi,x}\}_{x>0}$ is a Carleson measures family and the norm inequality between the spaces. $\square$

In a particular case one can give a partial converse to the above result.

Proposition 4.6.13. *Let* $p \geq 1$ *and* $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ *be an analytic function such that:*

(1) $C_\varphi : H_p^{(1)}(\mathbb{C}^+) \to H_p^{(1)}(\mathbb{C}^+)$ *is a bounded composition operator.*

(2) *There exists* $S > 0$ *such that*

$$\frac{|z|}{|\varphi(z)|} > S, \qquad \text{for all } z \in \mathbb{C}^+.$$

Then the family $\{\nu_{\varphi,M_1,x}\}_{x>0}$ *is a set of Carleson measures with common bound, where*

$$\nu_{\varphi,M_1,x}(E) = \int_{\{y \in \mathbb{R} \ : \ \varphi(x+iy) \in E\}} |\varphi'(x+iy)|^p dy, \qquad \text{for } E \in \mathcal{B}or(\mathbb{C}^+).$$

Proof. Let $p \geq 1$ and $f \in H_p^{(1)}(\mathbb{C}^+)$. Then

$$\begin{aligned}
\|C_\varphi f\|_{p,(1)}^p &= \sup_{x>0} \int_{-\infty}^{\infty} |f'(\varphi(x+iy))|^p |\varphi'(x+iy)|^p |x+iy|^p dy \\
&= \sup_{x>0} \int_{-\infty}^{\infty} |f'(\varphi(x+iy))|^p |\varphi'(x+iy)|^p |\varphi(x+iy)|^p \frac{|x+iy|^p}{|\varphi(x+iy)|^p} dy \\
&\geq \sup_{x>0} M^p \int_{-\infty}^{\infty} |f'(\varphi(x+iy))|^p |\varphi'(x+iy)|^p |\varphi(x+iy)|^p dy \\
&\geq M^p \int_{-\infty}^{\infty} |f'(\varphi(x_0+iy))|^p |\varphi'(x_0+iy)|^p |\varphi(x_0+iy)|^p dy \\
&= M^p \iint_{\mathbb{C}^+} |f'(\omega)|^p |\omega|^p d\nu_{\varphi,M_1,x_0}(\omega),
\end{aligned}$$

for each $x_0 > 0$. On the other hand, C_φ is a bounded operator and then

$$\|C_\varphi f\|_{p,(1)}^p \le C\|f\|_{p,(1)}^p = C\|f\|_p^p.$$

Finally we get that $\{\nu_{\varphi,M_1,x}\}_{x>0}$ is a family of Carleson measures with common bound. (See Theorem 3.9 in [G, p.61], for example). □

Note that the above argument does not work for general n in principle, because we need to split components of $|(C_\varphi f)^{(n)}(x+iy)|$ and give a bound from below for them.

The following result is a straightforward consequence of [M, Th. 2.8.].

Proposition 4.6.14. *Let $n \in \mathbb{N}$ and $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ an analytic function such that there exists $M > 0$ such that*

$$|\operatorname{Re}(\varphi')(\omega) + i\operatorname{Im}(\varphi')(z)| > M, \qquad \text{for all } \omega, z \in \mathbb{C}^+.$$

Then $\{\nu_{\varphi,x}\}_{x>0}$ are Carleson measures with common bound, where

$$\nu_{\varphi,x}(E) = |\{y \in \mathbb{R} \ : \ \varphi(x+iy) \in E\}|$$

for $E \in \mathcal{B}or(\mathbb{C}^+)$ and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

Corollary 4.6.15. *Let $n \in \mathbb{N}$ and $p \ge 1$. Let $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function such that*

$$\sup_{z\in\mathbb{C}^+, 1\le j\le n} |\varphi^{(j)}(z)\varphi^{j-1}(z)|$$

and there exists $S > 0$ such that

$$|\operatorname{Re}(\varphi')(\omega) + i\operatorname{Im}(\varphi')(z)| > S, \qquad \text{for all } \omega, z \in \mathbb{C}^+.$$

Then the family $\{\nu_{\varphi,M,x}\}_{x>0}$ is a set of Carleson measures with common bound, where

$$\nu_{\varphi,M,x}(E) = \int_{\{y\in\mathbb{R}\ :\ \varphi(x+iy)\in E\}} \Phi_{\varphi,M}(x+iy)dy$$

with $\Phi_{\varphi,M}(x+iy) = \prod_{j=1}^n |\varphi^{(j)}(x+iy)|^{m_j p}|\varphi(x+iy)|^{(j-1)m_j p}$ for $E \in \mathcal{B}or(\mathbb{C}^+)$ and each $M = (m_1, \cdots, m_n)$ satisfying (4.9).

Proof. Let $t \in \mathbb{R}$ and $x, h > 0$. From the first part of the proof of Proposition 4.6.14, $|y_1 - y_2| < \frac{\sqrt{2}}{M}h$ for all $y_1, y_2 \in \mathbb{R}$ such that $\varphi(x+iy_1), \varphi(x+iy_2) \in Q_{t,h}$. Then we get

$$\nu_{\varphi,M,x}(Q_{t,h}) \le C_{n,p}\int_{\{y\in\mathbb{R}\ :\ \varphi(x+iy)\in E\}} dy \le C_{n,p}\frac{\sqrt{2}}{S}h,$$

and $\{\nu_{\varphi,M,x}\}_{x>0}$ are Carleson measures with common bound. □

Corollary 4.6.16. *Let $n \in \mathbb{N}$ and $1 \leq p < \infty$. Let $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function such that*

$$\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty \qquad \text{and} \qquad \sup_{z\in\mathbb{C}^+, 1\leq j\leq n} |\varphi^{(j)}(z)\varphi^{j-1}(z)| < \infty$$

and there exists $M > 0$ such that

$$|\operatorname{Re}(\varphi')(\omega) + i\operatorname{Im}(\varphi')(z)| > M, \qquad \text{for all } \omega, z \in \mathbb{C}^+.$$

Then φ induces a bounded composition operator C_φ on $H_p^{(n)}(\mathbb{C}^+)$.

Proof. We can apply Corollary 4.6.15 and Theorem 4.6.11, or, alternatively, Proposition 4.6.14 and Corollary 4.6.12. □

Example 4.6.17. Note that the contrapositive of Corollary 4.6.9 states that, if φ does not induce a bounded composition operator C_φ on $H_2(\mathbb{C}^+)$, then φ can not induce a bounded operator on $H_2^{(n)}(\mathbb{C}^+)$. Then we use [M, Corollary 2.2 and Example 2.4] to affirm that neither a bounded φ nor $\varphi(z) = \sqrt{z}$ (where $\sqrt{|z|e^{i\theta}} = \sqrt{|z|}e^{i\frac{\theta}{2}}$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$) can induce a bounded operator on $H_2^{(n)}(\mathbb{C}^+)$.

For more results about boundedness of composition operators on $H_2^{(n)}(\mathbb{C}^+)$, see [GMMS]. There we prove that the following functions also induce a bounded composition operator on $H_2^{(n)}(\mathbb{C}^+)$.

Examples 4.6.18. (1) Consider the rational map

$$r(z) = \frac{a_r z^r + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}$$

with $a_r, b_m \neq 0$. In [E] it is proved that, if r is such that $r(\infty) = \infty$ and $r(\mathbb{C}^+) \subseteq \mathbb{C}^+$ then necessarily

(1) $r = m + 1$,

(2) $\frac{a_r}{b_m} \in \mathbb{R}$, and in particular, $\frac{a_r}{b_m} > 0$,

(3) $\operatorname{Im}(\frac{a_0}{b_0}) \geq 0$.

If we consider a map r such that $0 \notin r(i\mathbb{R})$ then the operator C_r is bounded on $H_p^{(n)}(\mathbb{C}^+)$. The same can be applied to

$$\rho(z) = \frac{a_1 z^{\alpha_1} + a_2 z^{\alpha_2} + \cdots + a_m z^{\alpha_m}}{b_1 z^{\beta_1} + b_2 z^{\beta_2} + \cdots + b_p z^{\beta_p}}$$

with $\alpha_i, \beta_j \neq 0$, not neccesarily integers, assuming $\alpha_i > \alpha_{i+1}$, $\beta_j > \beta_{j+1}$, and $\alpha_1 = \beta_1 + 1$.

(2) The map

$$\varphi(z) = az + b\sqrt{z} + c, \qquad z \in \mathbb{C}^+,$$

with $a, b > 0$ and $\operatorname{Re} c \geq 0$ the operator C_φ is bounded on $H_2^{(n)}(\mathbb{C}^+)$ for $n \geq 1$. The case $n = 0$ was proved in [M, Example 2.9].

Finally we consider $\varphi(z) = az + b\log(1 + z)$ with $a, b > 0$. Then the operator C_φ is bounded on $H_p^{(n)}(\mathbb{C}^+)$ for $n \geq 1$.

Chapter 5

Further generalizations of Cesàro-Hardy operators and range spaces

In this final chapter we approach some questions about generalizations of the operators and range spaces considered formerly, in two directions.

We first introduce generalized Cesàro-Hardy operators associated with locally integrable functions $\kappa \in L^1_{loc}(\mathbb{R}^+)$. These operators are of the form

$$\mathcal{C}_\kappa(f) := \frac{1}{\chi_{(0,\infty)} * \kappa} \kappa * f, \quad f \in L^1_{loc}(\mathbb{R}^+),$$

with dual (adjoint) operator

$$\mathcal{C}^*_\kappa(f) := \kappa \circ \left(\frac{f}{\chi_{(0,\infty)} * \kappa} \right)$$

where $(\kappa \circ g)(t) := \int_t^\infty \kappa(s-t) g(s) ds$, $t > 0$ for every $g \in L^1_{loc}(\mathbb{R}^+)$ such that the integral exists.

Sufficient conditions on κ are given for $\mathcal{C}_\kappa$ and $\mathcal{C}^*_\kappa$ to be bounded on $L_p(\mathbb{R}^+)$. Range spaces corresponding to operators $\mathcal{C}^*_\kappa$ are then defined and its convolution algebraic structure is discussed.

In a second direction, we focus on operators $\mathcal{C}^*_\kappa$ and their ranges in order to establish a framework with application to abstract Cauchy problems.

In 1833 J.M.C. Duhamel considered the following evolution problem corresponding to the initial-boundary value problem for the heat equation in a domain Ω (Ω is an open subset of $\mathbb{R}^n$):

$$\text{(5.1)} \qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u, & (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = u_0(x), & x \in \Omega; \\ u(t,\cdot)|_{\partial\Omega} = g(t,\cdot), & t > 0. \end{cases}$$

and proposed the following formula to express the solution of (5.1).

$$u(t,x) = \int_0^t \frac{\partial}{\partial t} u(\lambda, t-\lambda, x) d\lambda, \quad t > 0,$$

where $u(\lambda, t, x)$ is a solution of (5.1) for a particular function $g(\cdot, \lambda_0)$ with fixed λ_0 [D]. This formula allows one to reduce the Cauchy problem for an in-homogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation. This formula (known also as Duhamel's principle) is of widespread use in partial differential equations and has been studied in a large number of papers, see for example [DL, U]. In the paper [U], the author extend the Duhamel principle to fractional order equations.

Here we establish an abstract version of the Duhamel formula which allows us to extend local convoluted solutions of Cauchy problems (Theorem 5.2.17). In the way to do this, a new class of test function space is defined on which one can introduce a new class of κ-distribution semigroups on the basis of the convolution structure quoted above.

5.1 Algebraic structures defined by Cesàro-Hardy type operators

There exists a wide literature about weighted inequalities of Cesàro-Hardy type, as it can be seen in [KuP, KuMP, OK] and also [A, KNPW, MS]. Here we are going to deal with weighted inequalities of a very particular type. Note that the Cesàro-Hardy operator, for $\nu > 0$, given in (1.6) may be written in the following way:

$$\mathcal{C}_\nu f(t) := \frac{\nu\Gamma(\nu)}{t^\nu}\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} f(s)ds = \frac{1}{(\chi_{(0,\infty)} * \mathfrak{r}_\nu)(t)}(f * \mathfrak{r}_\nu)(t), \qquad t \geq 0,$$

and then the Hardy inequality (3) may be written as:

$$\|\frac{1}{\chi_{(0,\infty)} * \mathfrak{r}_\nu} f * \mathfrak{r}_\nu\|_p \leq \frac{\Gamma(\nu+1)\Gamma(1-\frac{1}{p})}{\Gamma(\nu+1-\frac{1}{p})}\|f\|_p, \qquad f \in L^p(\mathbb{R}^+),$$

for $1 < p < \infty$.

The above reformulations suggest extending Cesàro-Hardy operators to operators of the form

$$\mathcal{C}_\kappa(f) = \frac{1}{\chi_{(0,\infty)} * \kappa} f * \kappa, \qquad \mathcal{C}^*_\kappa(f) := \kappa \circ \left(\frac{f}{\chi_{(0,\infty)} * \kappa}\right), \tag{5.2}$$

for suitable functions f. Of course, a first question in this direction is that one of the boundedness of such operators.

In the first part of this section, we give sufficient conditions on κ for the operator $\mathcal{C}_\kappa$ to be bounded on $L_p(\mathbb{R}^+)$, see Theorem 5.1.5; that is, for $\mathcal{C}_\kappa$ to satisfy the inequality

$$\|\frac{1}{\chi_{(0,\infty)} * \kappa} f * \kappa\|_p \leq C_{\kappa,p}\|f\|_p, \qquad f \in L_p(\mathbb{R}^+), \quad 1 < p < \infty. \tag{5.3}$$

This kind of inequality can be seen as a weighted inequality for Hardy-Volterra integral operators, see [KuMP, Section 9.B] and [KNPW, Section 4].

The boundedness of operator $\mathcal{C}_\kappa$ (or its adjoint) is used in the second subsection to establish a result about the relationship of Banach modules and Banach algebras concerning range spaces of $\mathcal{C}_\kappa^*$, Theorem 5.1.15. In the third part of this section, we show examples of functions κ and corresponding spaces where to apply our results.

As an appendix of this section, we discuss the doubling condition and the Ariño-Muckenhoupt condition on weights, around the above circle of ideas.

5.1.1 Convolution and Hardy-type operators

Given $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ as a measurable function and $1 \le p < \infty$, consider $L_p(\omega^p)$, the set of weighted Lebesgue p-integrable functions f, whose norm will be denoted $\| \cdot \|_{p,\omega}$.

If $\omega(t+s) \le C\omega(t)\omega(s)$, a.e. for $t, s \ge 0$ and $C > 0$, then $f * g \in L_p(\omega)$ and

$$\|f * g\|_{p,\omega} \le C\|f\|_{1,\omega} \, \|g\|_{p,\omega},$$

for $f \in L_1(\omega)$ and $g \in L_p(\omega)$; if $\omega(t-s) \le C\omega(t)\omega(s)$ a.e. for $t \ge s \ge 0$ and $C > 0$, then $f \circ g \in L_p(\omega)$ and

$$\|f \circ g\|_{p,\omega} \le C\|f\|_{1,\omega} \, \|g\|_{p,\omega},$$

for $f \in L_1(\omega)$ and $g \in L_p(\omega)$, see [Mi3]. We show similar inequalities in the following straightforward proposition.

Proposition 5.1.1. *Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-negative and non-decreasing a.e. function and $1 \le p < \infty$. Then*

(1) $L_p(\mathbb{R}^+) \circ L_1(\omega) \hookrightarrow L_p(\omega)$;

(2) $L_1(\mathbb{R}^+) \circ L_p(\omega) \hookrightarrow L_p(\omega)$.

Definition 5.1.2. *Let $p \ge 1$ and let κ in $L^1_{loc}(\mathbb{R}^+)$ non-negative. Set $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$.*

(1) *We say that κ satisfies the (HC)$_p$ condition (Hardy-type condition) if there exists $C_{\kappa,p} > 0$ such that*

$$\|\kappa * g\|_{p,\frac{1}{\mathfrak{K}}} \le C_{\kappa,p}\|g\|_p, \qquad \text{for all } g \in L_p(\mathbb{R}^+). \tag{5.4}$$

(2) *We say that κ satisfies the (dHC)$_p$ condition (dual Hardy-type condition) if there exists $C'_{\kappa,p} > 0$ such that*

$$\|\kappa \circ f\|_p \le C'_{\kappa,p}\|f\|_{p,\mathfrak{K}}, \qquad \text{for all } f \in L_p(\mathfrak{K}^p). \tag{5.5}$$

For $p = \infty$, inequality (5.4) holds for any measurable and positive function $\kappa \in L^1_{loc}(\mathbb{R}^+)$. Similarly, for $p = 1$, inequality (5.5) holds for any measurable and positive function $\kappa \in L^1_{loc}(\mathbb{R}^+)$ (without additional conditions). However for $p = 1$ and $\kappa = \chi_{(0,\infty)}$, inequality (5.4) does not hold: take $g(y) = \chi_{(0,1)}(y)\frac{1}{\sqrt{y}}$.

The products $*$ and $\circ$ are dual convolution products in the following sense: the equality

$$\int_0^\infty (\kappa * f)(t)g(t)dt = \int_0^\infty (\kappa \circ g)(s)f(t)dt$$

holds for "good" functions κ, f and g.

Theorem 5.1.3. *Let κ be a nonnegative measurable function and let $1<p<\infty$. Then κ satisfies the (HC)$_p$ condition if and only if κ satisfies the (dHC)$_q$ condition for q the conjugate exponent of p.*

Proof. Suppose that κ satisfies the *(HC)$_p$* condition. Take $f\in L_q(\mathfrak{K})$ and let

$$\tilde f(x) := \int_x^\infty \kappa(y-x)f(y)dy = (\kappa\circ f)(x), \qquad x\geq 0.$$

Let $g\in L_p(\mathbb{R}^+)$. Then

$$\begin{aligned}
\|\tilde f g\|_1 &\leq \int_0^\infty \left(\int_0^y \kappa(y-x)|g(x)|dx\right)|f(y)|dy\\
&\leq \left(\int_0^\infty \frac{1}{\left(\int_0^y\kappa(\tau)d\tau\right)^p}\left(\kappa*|g|(y)\right)^p dy\right)^{\frac1p}\left(\int_0^\infty\left(\int_0^y\kappa(\tau)d\tau\right)^{p'}|f(y)|^q dy\right)^{\frac1q}\\
&\leq C_{\kappa,p}\|g\|_p\left(\int_0^\infty(\kappa*\chi_{(0,\infty)}(y)|f(y)|)^q dy\right)^{\frac1q} = C_{\kappa,p}\|g\|_p\|f\|_{q,\mathfrak{K}}
\end{aligned}$$

where Fubini's theorem as been applied in the first equality, Hölder's inequality in the second one and the *(HC)$_p$* condition in the third one. This implies that $\kappa\circ f\in L_q(\mathbb{R}^+)$, $\|\kappa\circ f\|_q\leq C_{\kappa,p}\|f\|_{q,\mathfrak{K}}$ and κ satisfies the *(dHC)$_q$* condition. Similarly, we prove the converse result. □

Examples 5.1.4. **(1)** It is clear that the function $\mathfrak{r}_\nu(t) := \frac{t^{\nu-1}}{\Gamma(\nu)}$ for $\nu>0$ and $\chi_{(0,\infty)}$ satisfies the *(HC)$_p$* and *(dHC)$_p$*, whereas characteristic function $\chi_{(1,\infty)}$ does not satisfy the $(HC)_p$ for any $1<p<\infty$.

(2) Exponential functions $\{e_\lambda\ : 0\neq\lambda\in\mathbb{R}\}$ (recall that $e_\lambda(t) := e^{-\lambda t}$ for $t>0$) do not satisfy the $(HC)_p$ for any $1<p<\infty$. In fact, we check that $\{e_\lambda\ : 0\neq\lambda\in\mathbb{R}\}$ do not satisfy the $(dHC)_p$ for any $1<p<\infty$. Take $\lambda,\mu>0$, we get that $e_\lambda\circ e_\mu = \frac{1}{\lambda+\mu}e_\mu$ and

$$\|e_\lambda\circ e_\mu\|_p = \frac{1}{\lambda+\mu}\|e_\mu\|_p = \frac{1}{\lambda+\mu}\left(\frac{1}{\mu p}\right)^{\frac1p}.$$

Note that $e_\lambda*\chi_{(0,\infty)}(t) = \frac{1-e^{-\lambda t}}{\lambda}$ for $t>0$ and

$$\|e_\mu\|^p_{p,e_\lambda*\chi_{(0,\infty)}} = \frac{1}{\lambda^p}\int_0^\infty e^{-\mu pt}(1-e^{-\lambda t})^p dt \leq \int_0^\infty e^{-\mu pt}t^p dt = \frac{\Gamma(p+1)}{(\mu p)^{p+1}}$$

for $\mu>0$. Then there does not exist $C>0$ such that

$$\frac{1}{\lambda+\mu}\left(\frac{1}{\mu p}\right)^{\frac1p} \leq C\,\frac{\Gamma(p+1)^{\frac1p}}{(\mu p)^{1+\frac1p}}$$

for every $\mu>0$.

Now take $\lambda > 0$ and $\mu > \lambda$. Then $e_{-\lambda} \circ e_\mu = \frac{1}{\mu-\lambda} e_\mu$ and $e_{-\lambda} * \chi_{(0,\infty)}(t) = \frac{1-e^{\lambda t}}{\lambda}$ for $t > 0$. Note that

$$\|e_\mu\|^p_{p,e_{-\lambda}*\chi_{(0,\infty)}} = \frac{1}{\lambda^p}\int_0^\infty e^{-\mu p t}(e^{\lambda t}-1)^p dt \leq \frac{1}{\lambda^p}\int_0^\infty e^{-(\mu-\lambda)pt} dt = \frac{1}{\lambda^p}\frac{1}{(\mu-\lambda)p},$$

and there does not exist $C' > 0$ such that

$$\frac{1}{\mu-\lambda}\left(\frac{1}{\mu p}\right)^{\frac{1}{p}} \leq C' \frac{1}{\lambda}\frac{1}{((\mu-\lambda)p)^{\frac{1}{p}}},$$

for any $\mu > \lambda$.

The next result is a particular case of [KNPW, Theorem 4.4]. Condition "$A_{\kappa,p}(r) < \infty$" is condition (4.7) given in [KNPW, Theorem 4.4] for $r \in (1,p)$. We have decided to include it here to avoid the lack of completeness but we remove the proof.

Proposition 5.1.5. *Let k be a nonnegative measurable function with $\int_0^\varepsilon \kappa(x)dx > 0$ for all $\varepsilon > 0$ and there exists $r \in \mathbb{R}$ such that*

$$\text{(5.6)} \qquad ess\, sup_{s\in(0,\infty)} s^{\frac{r-1}{p}} \left(\int_s^\infty \left(\frac{\kappa(u-s)}{\int_0^u \kappa(x)dx}\right)^p u^{p-r} du\right)^{\frac{1}{p}} =: A_{\kappa,p}(r) < \infty,$$

for some $p > 1$ and $1 < r < p$. Then

$$\|g * \kappa\|_{p,\frac{1}{\mathfrak{K}}} \leq A_{\kappa,p}(r)\|g\|_p, \qquad \text{for all } g \in L_p(\mathbb{R}^+),$$

where $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$, that is, the function κ satisfies the (HC)$_p$ condition and

$$\|\mathcal{C}_\kappa\|_{\mathcal{B}(L_p(\mathbb{R}^+))} \leq \inf_{1<r<p}\left(\left(\frac{p-1}{p-r}\right)^{\frac{1}{q}} A_{\kappa,p}(r)\right),$$

where $\mathcal{C}_\kappa(g) = \frac{1}{\chi_{(0,\infty)}*\kappa}(\kappa * g)$, for $g \in L_p(\mathbb{R}^+)$.

Note that the inequality (5.6) may be written in terms of $\circ$ product due to

$$\kappa^p \circ \left(\frac{h_{1-\frac{r}{p}}}{\chi_{(0,\infty)} * \kappa}\right)^p (s) = \int_s^\infty \left(\frac{\kappa(u-s)}{\int_0^u \kappa(x)dx}\right)^p u^{p-r} du, \qquad s \geq 0,$$

and $h_{1-\frac{r}{p}}(u) = u^{1-\frac{r}{p}}$ for $u > 0$. For the shake of completeness, we include some properties of the function $A_{\kappa,p}(r)$.

Lemma 5.1.6. *Take κ such that $A_{\kappa,p}(r) < \infty$ for some $p > 1$ and $1 < r < p$. Then*

(1) $A_{\kappa,p}(r') \leq A_{\kappa,p}(r)$ *for $p > r' > r > 1$;*

(2) $A_{\kappa,s}(r) \leq \frac{A_{\kappa,p}(r)}{(r-1)^{\frac{p-s}{sp}}}$ *for $1 < s < p$ and $r > 1$.*

Proof. (1) Take $r' > r$, and we get

$$s^{\frac{r'-1}{p}} \left(\int_s^\infty \left(\frac{\kappa(u-s)}{\int_0^u \kappa(x)dx} \right)^p u^{p-r'} du \right)^{\frac{1}{p}} \leq s^{\frac{r-1}{p}} \left(\int_s^\infty \left(\frac{\kappa(u-s)}{\int_0^u \kappa(x)dx} \right)^p x^{r'-r} u^{p-r'} du \right)^{\frac{1}{p}}$$

for $s > 0$, and then $A_{\kappa,p}(r') \leq A_{\kappa,p}(r)$. To show the part (2), take the pair of conjugate exponents $(\frac{p}{s}, \frac{p}{p-s})$ with $p > s$ and apply the Hölder inequality, as follows:

$$\int_t^\infty \left(\frac{\kappa(u-t)}{\int_0^u \kappa(x)dx} \right)^s u^{s-r} du \leq \left(\int_t^\infty \left(\frac{\kappa(u-t)}{\int_0^u \kappa(x)dx} \right)^p u^{p-r} du \right)^{\frac{s}{p}} \left(\int_t^\infty \frac{1}{u^r} du \right)^{\frac{p-s}{p}}$$

and then

$$t^{\frac{r-1}{s}} \left(\int_t^\infty \left(\frac{\kappa(u-t)}{\int_0^u \kappa(x)dx} \right)^s u^{s-r} du \right)^{\frac{1}{s}} \leq \frac{t^{\frac{r-1}{p}}}{(r-1)^{\frac{p-s}{sp}}} \left(\int_t^\infty \left(\frac{\kappa(u-t)}{\int_0^u \kappa(x)dx} \right)^p u^{p-r} du \right)^{\frac{1}{p}}$$

for $t > 0$. We conclude that $A_{\kappa,q}(r) \leq \frac{A_{\kappa,p}(r)}{(r-1)^{\frac{p-s}{sp}}}$. □

Examples 5.1.7. (1) Let κ be a function for which it is possible to find constants $0 < m \leq M$ and $\nu > 0$ such that

$$m\mathfrak{r}_\nu(t) \leq \kappa(t) \leq M\mathfrak{r}_\nu(t). \tag{5.7}$$

We get

$$\begin{aligned} s^{\frac{r-1}{p}} \left(\int_s^\infty \nu^p \frac{(u-s)^{(\nu-1)p}}{u^{\nu p}} u^{p-r} du \right)^{\frac{1}{p}} &= \nu \left(\int_0^\infty \frac{x^{(\nu-1)p}}{(1+x)^{\nu p - p + r}} dx \right)^{\frac{1}{p}} \\ &= \nu \left(\frac{\Gamma((\nu-1)p+1)\Gamma(r-1)}{\Gamma(\nu p - p + r)} \right)^{\frac{1}{p}} \end{aligned}$$

and κ satisfies condition (5.6) for $r > 1$ and $1 \leq p \leq \frac{1}{1-\nu}$ when $0 < \nu < 1$; κ satisfies condition (5.6) for $r > 1$ and $1 \leq p$ when $\nu \geq 1$. In all this cases, we obtain

$$\frac{m}{M} \nu \left(\frac{\Gamma((\nu-1)p+1)\Gamma(r-1)}{\Gamma(\nu p - p + r)} \right)^{\frac{1}{p}} \leq A_{\kappa,p}(r) \leq \frac{M}{m} \nu \left(\frac{\Gamma((\nu-1)p+1)\Gamma(r-1)}{\Gamma(\nu p - p + r)} \right)^{\frac{1}{p}}.$$

In fact condition (5.7) implies that the function κ may be written as $\kappa = h\mathfrak{r}_\nu$, where $h \in L_\infty(\mathbb{R}^+)$ and $\inf_{t \geq 0} h(t) \gneqq 0$; then $m = \inf_{t \geq 0} h(t)$ and $M = \|h\|_\infty$. Particular cases are

(a) the trivial case $\kappa := \mathfrak{r}_\nu$ for $\nu > 0$ and $A_{\mathfrak{r}_\nu,p}(r) = \nu \left(\frac{\Gamma((\nu-1)p+1)\Gamma(r-1)}{\Gamma(\nu p - p + r)} \right)^{\frac{1}{p}}$.

(b) the family $\kappa(t) := \left(\frac{At+B}{Ct+D} \right)^\gamma \mathfrak{r}_\nu(t)$, for $A, B, C, D, \nu, \gamma > 0$. In this case,

(∗) if $AD - BC < 0$, then $m = \left(\frac{A}{C}\right)^\gamma$ and $M = \left(\frac{B}{D}\right)^\gamma$.

(∗) if $AD - BC = 0$, then $\kappa = \left(\frac{B}{D}\right)^\gamma \mathfrak{r}_\nu$.

(∗) if $AD - BC > 0$, then $m = \left(\frac{B}{D}\right)^\gamma$ and $M = \left(\frac{A}{C}\right)^\gamma$.

(2) Let $\lambda > 0$ and consider functions e_λ, $e_{-\lambda}$. Then $A_{e_\lambda,p}(p) = A_{e_{-\lambda},p}(p) = \infty$ for $p \geq 1$: take $s > 0$ and consider

$$s^{1-\frac{1}{p}} \left(\int_s^\infty \frac{e^{\lambda(u-s)p}}{\left(\int_0^u e^{\lambda r} dr\right)^p} du \right)^{\frac{1}{p}} = \lambda s^{1-\frac{1}{p}} e^{-\lambda s} \left(\int_s^\infty \frac{1}{(1-e^{-\lambda u})^p} du \right)^{\frac{1}{p}} = \infty,$$

and

$$\lim_{s\to\infty} s^{1-\frac{1}{p}} \left(\int_s^\infty \frac{e^{-\lambda(u-s)p} du}{\left(\int_0^u e^{-\lambda r} dr\right)^p} \right)^{\frac{1}{p}} = \lim_{s\to\infty} s^{1-\frac{1}{p}} \lambda \left(\int_0^\infty \frac{e^{-\lambda x p}}{\left(1-e^{-\lambda(x+s)}\right)^p} dx \right)^{\frac{1}{p}} = \infty.$$

(3) The characteristic function $\chi_{(0,1)}$ satisfies the assumption that $A_{\chi_{(0,1)},p}(p) = \infty$ for $p \geq 1$, such that

$$\sup_{s\geq 1} \left(s^{1-\frac{1}{p}} \left(\int_s^\infty \frac{\chi_{(0,1)}(u-s)}{\left(\int_0^u \chi_{(0,1)}(r) dr\right)^p} du \right)^{\frac{1}{p}} \right) = \sup_{s\geq 1} \left(s^{1-\frac{1}{p}} \left(\int_s^{s+1} du \right)^{\frac{1}{p}} \right) = \infty.$$

Note that the characteristic function $\chi_{(1,\infty)}$ verifies $\int_0^\varepsilon \chi_{(1,\infty)}(s) ds = 0$ for $0 < \varepsilon \leq 1$.

The next theorem provides the boundedness of the operator of $f \mapsto \kappa \circ f$ between L_p-spaces. Similar results can be found in the literature, for example in [KNPW, Theorem 4.3].

Theorem 5.1.8. *Let κ be a non-negative measurable function with $\int_0^\varepsilon \kappa(x) dx > 0$ for all $\varepsilon > 0$ and there exists $r \in \mathbb{R}$ such that*

$$\text{(5.8)} \qquad ess\, sup_{s\in(0,\infty)} \frac{s^{\frac{r-1}{p}}}{\int_0^s \kappa(x) dx} \left(\int_0^s \kappa^p(s-u) u^{p-r} du \right)^{\frac{1}{p}} =: B_{\kappa,p}(r) < \infty,$$

for some $p > 1$ and $p + 1 > r > p$. Then

$$\|\kappa \circ f\|_p \leq B_{\kappa,p}(r) \|f\|_{p,\mathfrak{K}} \qquad \text{for all } f \in L_p(\mathbb{R}^+),$$

where $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$, that is, the function κ satisfies the $(dHC)_p$ condition and

$$\|T'_\kappa\|_{\mathcal{B}(L_p(\mathbb{R}^+), L_p(\mathbb{R}^+))} \leq \inf_{p+1>r>p} \left(\left(\frac{p-1}{r-p} \right)^{\frac{1}{q}} B_{k,p}(r) \right),$$

where $T'_\kappa f := \kappa \circ f$.

Proof. Take $f \in L_p(\mathfrak{K}^p)$ and then

$$\begin{aligned}\|\kappa \circ f\|_p &\leq \left(\int_0^\infty \left(\int_x^\infty \kappa(s-x)|f(s)|ds\right)^p dx\right)^{\frac{1}{p}} = \left\|\int_1^\infty \kappa(x(t-1))|f(xt)|xdt\right\|_p \\ &\leq \int_1^\infty \|\kappa(x(t-1))|f(xt)|x\|_p dt = \int_1^\infty \left(\int_0^\infty (\kappa(x(t-1))|f(xt)|x)^p\, dx\right)^{\frac{1}{p}} dt,\end{aligned}$$

where we change the variable $s = xt$ and we apply Minkowski's integral inequality. Take q as the conjugate exponent of p and apply the Hölder inequality and Fubini's theorem to get

$$\begin{aligned}\|\kappa \circ f\|_p &\leq \int_1^\infty t^{-\frac{r-1}{p}} t^{\frac{r-1}{p}} \left(\int_0^\infty (\kappa(x(t-1))|f(xt)|x)^p\, dx\right)^{\frac{1}{p}} dt \\ &\leq \left(\int_1^\infty t^{-\frac{(r-1)q}{p}} dt\right)^{\frac{1}{q}} \left(\int_0^\infty |f(s)|^p s^{r-1} \int_0^s (\kappa(s-u))^p u^{p-r} du ds\right)^{\frac{1}{p}} \\ &\leq \left(\frac{p-1}{r-p}\right)^{\frac{1}{q}} B_{\kappa,p}(r) \left(\int_0^\infty |f(s)|^p \left(\int_0^s \kappa(x)dx\right)^p ds\right)^{\frac{1}{p}},\end{aligned}$$

where we have changed the variable and applied the assumption that κ satisfies (5.8). We conclude that

$$\|T'_\kappa\|_{\mathcal{B}(L_p(\mathbb{R}^+))} \leq \inf_{p+1>r>p} \left(\left(\frac{p-1}{r-p}\right)^{\frac{1}{q}} B_{\kappa,p}(r)\right),$$

and the theorem is shown. □

Remark 5.1.9. Note that $T'_\kappa f = \mathcal{C}^*_\kappa\left(\frac{f}{\mathfrak{K}}\right)$. And therefore the boundedness of T'_κ is equivalent to the boundedness of $\mathcal{C}^*_\kappa$. We use T'_κ just to clarify some of the calculations.

To finish this subsection we present a table where it may be found functions and its behavior with respect to several conditions considered in this section (condition *(HC)*$_p$ and *(dHC)*$_p$) and in subsection 5.1.4 (conditions *(DC)*, *(DIC)* and *(AMC)*$_p$):

function \ condition	*(HC)*$_p$	*(dHC)*$_p$	*(DC)*	*(DIC)*	*(AMC)*$_p$
$\mathfrak{r}_\nu$	$p > 1$	$p \geq 1$	✓	✓	$p > 1$
$\chi_{(0,1)}$	$p > 1$	$p \geq 1$	✓	✓	$p > 1$
e_λ	×	×	✓	✓	$p > 1$
$e_{-\lambda}$	×	×	×	✓	×
$\chi_{(1,\infty)}$	×	×	×	×	×

for $\nu, \lambda > 0$.

5.1.2 Convolution Banach modules $\mathcal{T}_p^\kappa(\mathbb{R}^+)$

In the beginning of this subsection we collect some definitions and properties that will be used throughout this section. We will denote by $\mathcal{D}_+$ the set $C_c^{(\infty)}[0,\infty)$. Note that the condition $0 \in \operatorname{supp}(\kappa)$ is equivalent to suppose that the function κ is not identically zero on $[0,\varepsilon)$ for all $\varepsilon > 0$.

Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be such that $0 \in \operatorname{supp}(\kappa)$. We define the operator $T'_\kappa : \mathcal{D}_+ \to \mathcal{D}_+$ given by $f \mapsto T'_\kappa(f) := \kappa \circ f$.

(1) Then $T'_\kappa : \mathcal{D}_+ \to \mathcal{D}_+$ is an injective, linear and continuous homomorphism such that
$$T'_\kappa(f \circ g) = f \circ T'_\kappa(g), \qquad f, g \in \mathcal{D}_+.$$

(2) The map T'_κ extends to a linear and continuous map from $L_1(\mathfrak{K})$ to $L_1(\mathbb{R}^+)$, which we denote again by $T'_\kappa : L_1(\mathfrak{K}) \to L_1(\mathbb{R}^+)$ such that $\|T'_\kappa\| \le 1$.

See [KLM, Theorem 2.5]. Then, we define the space $\mathcal{D}_\kappa$ by $\mathcal{D}_\kappa := T'_\kappa(\mathcal{D}_+)$ and the map $W_\kappa : \mathcal{D}_\kappa \to \mathcal{D}_+$ by
$$f(t) = T'_\kappa(W_\kappa(f))(t) = \int_t^\infty \kappa(s-t) W_\kappa f(s)\,ds, \qquad f \in \mathcal{D}_\kappa, \quad t \ge 0,$$
see [KLM] for more details.

Examples 5.1.10. (1) As we have stated many times along the monograph, if we take $\nu > 0$ and $\kappa = \mathfrak{r}_\nu$; the map $W_{\mathfrak{r}_\nu}$ is the Weyl fractional derivative of order ν, W^ν, and $\mathcal{D}_{\mathfrak{r}_\nu} = \mathcal{D}_+$.

(2) Given $\nu > 0$ and $z \in \mathbb{C}$, take $\kappa = e_z \mathfrak{r}_\nu$, we have that $\mathcal{D}_\kappa = \mathcal{D}_+$ and
$$W_{e_z \mathfrak{r}_\nu} f = e_z W^\nu(e_{-z} f), \qquad f \in \mathcal{D}_+.$$
See other examples in [KLM, Section 2].

(3) For $\kappa = \chi_{(0,1)}$, it is straightforward to check that $T'_{\chi_{(0,1)}}(f)(t) = \int_t^{t+1} f(s)\,ds$ for $f \in \mathcal{D}_+$, $\mathcal{D}_{\chi_{(0,1)}} = \mathcal{D}_+$ and
$$W_{\chi_{(0,1)}} f(t) = -\sum_{n=0}^\infty f'(t+n), \qquad f \in \mathcal{D}_+, \quad t \ge 0.$$

Take $f, g \in \mathcal{D}_\kappa$. Then $f * g$, $f \circ g$, $f *_c g \in \mathcal{D}_\kappa$ and

$$(5.9) \quad W_\kappa(f*g)(s) = \int_0^s W_\kappa g(r) \int_{s-r}^s \kappa(t+r-s) W_\kappa f(t)\,dt\,dr - \int_s^\infty W_\kappa g(r) \int_s^\infty \kappa(t+r-s) W_\kappa f(t)\,dt\,dr;$$

$W_\kappa(f \circ g) = f \circ W_\kappa g$; and $W_\kappa(f *_c g) = \frac{1}{2}\left(W_\kappa(f*g) + f \circ W_\kappa g + g \circ W_\kappa f\right)$, see [KLM, Theorem 2.10].

We now extend the analog of the Sobolev Banach algebras to the corresponding spaces for $\kappa \in L^1_{loc}(\mathbb{R}^+)$.

Theorem 5.1.11. *(see [KLM, Theorems 3.4 and 3.5]) Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in$ supp(κ), satifies the doubling condition (DC) and $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$. Then the integral*

$$\|f\|_{1,\mathfrak{K}} := \int_0^\infty |W_\kappa f(t)|\mathfrak{K}(t)dt, \qquad f \in \mathcal{D}_\kappa$$

defines an algebra norm on $\mathcal{D}_\kappa$ for the convolution product $*$, and also for $*_c$. We denote by $\mathcal{T}^\kappa(\mathbb{R}^+)$ (or $\mathcal{T}_1^\kappa(\mathbb{R}^+)$) the Banach space obtained as the completion of $\mathcal{D}_\kappa$ in the norm $\|\cdot\|_{1,\mathfrak{K}}$, and then we have $\mathcal{T}^\kappa(\mathbb{R}^+) \hookrightarrow L_1(\mathbb{R}^+)$.

These Banach algebras $\mathcal{T}^\kappa(\mathbb{R}^+)$ are the algebras for which we want to stablish the module versus algebra relation. If they are somehow the analogues of $L_1(\mathbb{R}^+)$, we are going to define the Banach spaces that will act as the analogues of $L_p(\mathbb{R}^+)$, but we need some tools to do this construction.

From now on, we consider $\kappa \in L^1_{loc}(\mathbb{R}^+)$ as a nonnegative function such that $0 \in$ supp(κ) and $\mathfrak{K} = \kappa * \chi_{(0,\infty)}$. Let $1 < p < \infty$ and suppose that κ verifies the *(dHC)$_p$* condition. Take $F \in L_p(\mathfrak{K}^p)$. The function $T'_\kappa F$ given by

$$T'_\kappa F(t) = (\kappa \circ F)(t) = \int_t^\infty \kappa(s-t)F(s)ds, \qquad \text{a.e.} \quad t \geq 0,$$

belongs to $L_p(\mathbb{R}^+)$; moreover T'_κ is a bounded operator $T'_\kappa : L_p(\mathfrak{K}^p) \to L_p(\mathbb{R}^+)$, which extends the operator $T'_\kappa : \mathcal{D}_+ \to \mathcal{D}_+$.

Definition 5.1.12. *Let $\mathcal{T}_p^\kappa(\mathbb{R}^+)$ denote the Banach space formed as the set $T'_\kappa(L_p(\mathfrak{K}^p))$ endowed with the norm $|||\cdot|||_{p,\kappa}$ obtained as the image of the norm $\|\cdot\|_{p,\mathfrak{K}}$ of $L_p(\mathfrak{K}^p)$ through the operator $T'_\kappa : L_p(\mathfrak{K}^p) \to L_p(\mathbb{R}^+)$. For $p = 1$, we keep the notation $\mathcal{T}^\kappa(\mathbb{R}^+)$.*

In accordance with Definition 5.1.12, $T'_\kappa : L_p(\mathfrak{K}^p) \to \mathcal{T}_p^\kappa(\mathbb{R}^+)$ is a surjective isometry and $\mathcal{T}_p^\kappa(\mathbb{R}^+)$ is a Banach space. Let $W_\kappa : \mathcal{T}_p^\kappa(\mathbb{R}^+) \to L_p(\mathfrak{K}^p)$ be the inverse isometry of T'_κ and $W_\kappa : \mathcal{T}_p^\kappa(\mathbb{R}^+) \to L_p(\mathfrak{K}^p)$ extends the operator $W_\kappa : \mathcal{D}_\kappa \to \mathcal{D}_+$ defined in the beginning of this subsection. Note that given a function $f \in \mathcal{T}_p^\kappa(\mathbb{R}^+)$, then $f \in L_p(\mathbb{R}^+)$ and there exists a unique element in $L_p(\mathfrak{K}^p)$ (we denote by $W_\kappa f$) such that

$$f(x) = T'_\kappa(W_\kappa f)(x) = \int_x^\infty \kappa(y-x)W_\kappa f(y)dy, \qquad \text{a.e. } x \geq 0.$$

Then for every $f \in \mathcal{T}_p^\kappa(\mathbb{R}^+)$, the norm is given by

$$|||f|||_{p,\kappa} = \left(\int_0^\infty |W_\kappa f(t)|^p \mathfrak{K}^p(t)dt\right)^{\frac{1}{p}}.$$

With these ideas, it is easy to show that the continuous inclusion $\mathcal{T}_p^\kappa(\mathbb{R}^+) \hookrightarrow L_p(\mathbb{R}^+)$ holds.

Examples 5.1.13. **(1)** Again, for $\kappa = \mathfrak{r}_\nu$, we write $\mathcal{T}_p^{(\nu)}(t^\nu)$ instead of $\mathcal{T}_p^{\mathfrak{r}_\nu}(\mathbb{R}^+)$, for $1 \leq p < \infty$ and the norm $|||f|||_{p,\mathfrak{r}_\nu}$ is that given in Proposition 1.3.2 (4). As we have said before, these families of spaces may be considered as Sobolev spaces of fractional order. There are a huge literature about this topic, we only mention the monographs [SKM, RS] and reference therein. However, the result about the module algebra of $\mathcal{T}^p_{(\nu)}(\mathbb{R}^+)$ for $p \geq 1$ seems to be new, see Corollary 5.1.17.

(2) In the case $\kappa = e_\lambda \mathfrak{r}_\nu$, with $\nu, \lambda > 0$ and $p \geq 1$, we obtain the Banach space $\mathcal{T}_p^{e_\lambda \mathfrak{r}_\nu}(\mathbb{R}^+)$ embedded with the norm

$$|||f|||_{p,e_\lambda\mathfrak{r}_\nu} := \frac{1}{\Gamma(\nu)}\left(\int_0^\infty |W^\nu(e_{-\lambda}f)(t)|^p \left(\int_0^t s^{\nu-1}e^{-\lambda(s+t)}ds\right)^p dt\right)^{\frac{1}{p}}.$$

(3) Take $\kappa = \chi_{(0,1)}$ and $\mathfrak{K}(t) = \int_0^t \chi_{(0,1)}(s)ds = t\chi_{(0,1)}(t) + \chi_{[1,\infty)}(t)$, for $t \geq 0$. We obtain the Banach space $\mathcal{T}_p^{\chi_{(0,1)}}(\mathbb{R}^+)$ for $1 \leq p < \infty$ embedded with the norm

$$|||f|||_{p,\chi_{(0,1)}} := \left(\int_0^1 |\sum_{n=0}^\infty f'(t+n)|^p t^p dt + \int_1^\infty |\sum_{n=0}^\infty f'(t+n)|^p dt\right)^{\frac{1}{p}},$$

for $f \in \mathcal{D}_+$.

As an easy consequence of Proposition 5.1.1 and from the embedding $\mathcal{T}_p^\kappa(\mathbb{R}^+) \hookrightarrow L_p(\mathbb{R}^+)$ for $p \geq 1$, we get the next corollary.

Corollary 5.1.14. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a nonnegative function such that $0 \in supp(\kappa)$ and satisfies the Hardy-type condition (dHC)$_p$ for some $1 \leq p < \infty$. Then*

(1) $L_p(\mathbb{R}^+) \circ \mathcal{T}^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}_p^\kappa(\mathbb{R}^+)$, *and then* $\mathcal{T}_p^\kappa(\mathbb{R}^+) \circ \mathcal{T}^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}_p^\kappa(\mathbb{R}^+)$;

(2) $L_1(\mathbb{R}^+) \circ \mathcal{T}_p^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}_p^\kappa(\mathbb{R}^+)$, *and then* $\mathcal{T}^\kappa(\mathbb{R}^+) \circ \mathcal{T}_p^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}_p^\kappa(\mathbb{R}^+)$.

Now we set the main result of this section.

Theorem 5.1.15. *Let $1 < p < \infty$, κ satisfying (HC)$_q$, (DC) and (AMC)$_q$, for q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(1) $\mathcal{T}^\kappa(\mathbb{R}^+) * \mathcal{T}_p^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}_p^\kappa(\mathbb{R}^+)$;

(2) $\mathcal{T}^\kappa(\mathbb{R}^+) *_c \mathcal{T}_p^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}_p^\kappa(\mathbb{R}^+)$.

Proof. (1) Let $f, g \in \mathcal{D}_\kappa$. According to (5.9),

$$|W_\kappa(f*g)(s)|^p \leq C\left(\int_0^s |W_\kappa g(r)| \int_{s-r}^s \kappa(t+r-s)|W_\kappa f(t)|dtdr\right)^p$$
$$+ C\left(\int_s^\infty |W_\kappa g(r)| \int_s^\infty \kappa(t+r-s)|W_\kappa f(t)|dtdr\right)^p.$$

Therefore,

$$|||f*g|||_{p,\kappa} = \left(\int_0^\infty |W_\kappa(f*g)(s)|^p \mathfrak{K}^p(s)ds\right)^{\frac{1}{p}} \leq C(I+J),$$

where

$$I := \left(\int_0^\infty \mathfrak{K}^p(s)\left(\int_0^s |W_\kappa g(r)| \int_{s-r}^s \kappa(t+r-s)|W_\kappa f(t)|dtdr\right)^p ds\right)^{\frac{1}{p}},$$

$$J := \left(\int_0^\infty \mathfrak{K}^p(s)\left(\int_0^s |W_\kappa g(r)| \int_s^\infty \kappa(t+r-s)|W_\kappa f(t)|dtdr\right)^p ds\right)^{\frac{1}{p}}.$$

By Minkowski's integral inequality, we get

$$I \leq \int_0^\infty |W_\kappa g(r)|(I_1 + I_2)dr,$$

where

$$I_1 := \left(\int_r^{2r} \mathfrak{K}^p(s)\left(\int_{s-r}^s \kappa(t+r-s)|W_\kappa f(t)|dt\right)^p ds\right)^{\frac{1}{p}},$$

$$I_2 := \left(\int_{2r}^\infty \mathfrak{K}^p(s)\left(\int_{s-r}^s k(t+r-s)|W_\kappa f(t)|dt\right)^p ds\right)^{\frac{1}{p}}.$$

We apply Theorems 5.1.20 and 5.1.27 to complete the proof and get

$$|||f * g|||_{p,\kappa} \leq C|||f|||_{p,\kappa}|||g|||_{1,\kappa}.$$

(2) We use the definition of $*_c$, (1) and Corollary 5.1.14. □

Remark 5.1.16. For $p = 1$, $\kappa \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in \mathrm{supp}(\kappa)$ and verifying the *(DC)* condition, the following embeddings

$$\mathcal{T}^\kappa(\mathbb{R}^+) * \mathcal{T}^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}^\kappa(\mathbb{R}^+), \qquad \mathcal{T}^\kappa(\mathbb{R}^+) *_c \mathcal{T}^\kappa(\mathbb{R}^+) \hookrightarrow \mathcal{T}^\kappa(\mathbb{R}^+)$$

hold, see Theorem 5.1.11. Note that the condition *(dHC)*$_1$ and Theorem 5.1.27 hold for $p = 1$.

5.1.3 Examples, applications and remarks

In this subsection we apply the main theorem of this section, Theorem 5.1.15, to several particular examples of functions κ which have appeared before. We also give some remarks and comments.

Weighted fractional Sobolev spaces

Corollary 5.1.17. *The Banach space $\mathcal{T}_p^{(\nu)}(\mathbb{R}^+)$ is a module for the algebra $\mathcal{T}_1^{(\nu)}(\mathbb{R}^+)$ (with usual convolution $*$ or the cosine convolution $*_c$) for $1 < p < \infty$.*

Scattering Sobolev spaces

Take the function $\chi_{(0,1)}$,

$$W_{\chi_{(0,1)}}f(t) = -\sum_{n=0}^{\infty} f'(t+n), \qquad f \in \mathcal{D}_+, \quad t \geq 0,$$

and we consider the Banach space $\mathcal{T}_p^{\chi_{(0,1)}}(\mathbb{R}^+)$ for $1 \leq p < \infty$ embedded with the norm

$$|||f||_{p,\chi_{(0,1)}} := \left(\int_0^1 |\sum_{n=0}^{\infty} f'(t+n)|^p t^p dt + \int_1^{\infty} |\sum_{n=0}^{\infty} f'(t+n)|^p dt \right)^{\frac{1}{p}},$$

for $f \in \mathcal{D}_+$.

Corollary 5.1.18. *The Banach space $\mathcal{T}_p^{\chi_{(0,1)}}(\mathbb{R}^+)$ is a module for the algebra $\mathcal{T}_1^{\chi_{(0,1)}}(\mathbb{R}^+)$ (with the usual convolution $*$ or the cosine convolution $*_c$) for $1 < p < \infty$.*

Final Comments

Under some conditions of a nonnegative function $\kappa \in L^1_{loc}(\mathbb{R}^+)$, we have introduced some function spaces which are Banach modules (for the usual and cosine convolution product) with respect to certain function Banach algebras. Now we comment other points which might be considered in further studies, and we wish to mention here:

(1) For $p = 2$, the Banach space $\mathcal{T}_2^{\kappa}(\mathbb{R}^+)$ could be, in fact, a Hilbert space with the inner product

$$(f|g)_{2,\kappa} := \int_0^{\infty} W_\kappa f(t)\overline{W_\kappa g(t)} \left(\int_0^t \kappa(s)ds \right)^2 dt, \qquad f, g \in \mathcal{T}_2^{\kappa}(\mathbb{R}^+).$$

(2) For $1 < p < \infty$ the dual Banach space of $\mathcal{T}_p^{\kappa}(\mathbb{R}^+)$ may be written by $\mathfrak{T}_q^{\kappa}(\mathbb{R}^+)$ embedded with the norm

$$||||f||||_{q,\kappa} := \left(\int_0^{\infty} \left| \frac{\kappa * f(t)}{\int_0^t \kappa(s)ds} \right|^q dt \right)^{\frac{1}{q}}, \qquad f \in \mathfrak{T}_q^{\kappa}(\mathbb{R}^+),$$

where q is the conjugate exponent of p.

(3) It seems to be natural that reflexivity and interpolation properties hold in Banach spaces $\mathcal{T}_p^{\kappa}(\mathbb{R}^+)$ for $1 < p < \infty$.

5.1.4 Some geometric conditions and Lebesgue norm inequalities

The doubling condition

Let κ be a nonnegative measurable function. We say that κ satisfies *(DC)* (the doubling condition) if there exists $D_\kappa > 0$ such that

$$\text{(DC)} \qquad \int_0^{2t} \kappa(s)ds \le D_\kappa \int_0^t \kappa(s)ds, \qquad t \ge 0.$$

This condition is well-known in real analysis and measure theory. Note that $(\mathfrak{r}_\nu)_{\nu>0}$, $\chi_{(0,1)}$ or κ a nonincreasing function (in particular $(e_\lambda)_{\lambda>0}$) satisfy the doubling condition. However, the functions $(e_{-\lambda})_{\lambda>0}$ and $\chi_{(1,\infty)}$ do not satisfy *(DC)*.

Lemma 5.1.19. *Let κ be a nonnegative measurable function such that $\kappa \in L_1(\mathbb{R}^+)$, $\int_0^\varepsilon \kappa(t)dt > 0$ for all $\varepsilon > 0$ and there exists*

$$\lim_{\varepsilon \to 0^+} \frac{\int_0^{2\varepsilon} \kappa(s)ds}{\int_0^\varepsilon \kappa(s)ds}.$$

Then κ satisfies (DC), in particular $\mathfrak{r}_\nu e_\lambda$ satisfies (DC) for $\nu > 0$ and $\lambda \ge 0$.

Proof. Define $F(t) := \frac{\int_0^{2t} \kappa(s)ds}{\int_0^t \kappa(s)ds}$ for $t > 0$. Note that F is continuous in $(0, \infty)$, $\lim_{t\to\infty} F(t) = 1$ and there exists $\lim_{t\to 0^+} F(t)$. We conclude that κ satisfies the *(DC)* condition. $\square$

Theorem 5.1.20. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ a non-negative function which satisfies (DC) and $(dHC)_p$ for some $p \ge 1$ and $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$. Then there exists $C > 0$ such that*

$$(1) \qquad \left(\int_r^{2r} \mathfrak{K}^p(s) \left(\int_{s-r}^s \kappa(t + r - s)|f(t)|dt \right)^p ds \right)^{\frac{1}{p}} \le C\mathfrak{K}(r)\|f\|_{p,\mathfrak{K}};$$

$$(2) \qquad \left(\int_{2r}^{\infty} \mathfrak{K}^p(s) \left(\int_{s-r}^s \kappa(t + r - s)|f(t)|dt \right)^p ds \right)^{\frac{1}{p}} \le C\mathfrak{K}(r)\|f\|_{p,\mathfrak{K}},$$

for $r \ge 0$ and $f \in L_p(\mathfrak{K}^p)$.

Proof. (1) Let $I_1 := \left(\int_r^{2r} \mathfrak{K}^p(s) \left(\int_{s-r}^s \kappa(t + r - s)|f(t)|dt \right)^p ds \right)^{\frac{1}{p}}$. Then we use *(DC)* to get that

$$\begin{aligned} I_1 &\le C \left(\int_r^{2r} \mathfrak{K}^p(r) \left(\int_{s-r}^s \kappa(t + r - s)|f(t)|dt \right)^p ds \right)^{\frac{1}{p}} \\ &= C\mathfrak{K}(r) \left(\int_0^r \left(\int_x^{x+r} \kappa(t - x)|f(t)|dt \right)^p dx \right)^{\frac{1}{p}} \\ &\le C\mathfrak{K}(r) \left(\int_0^\infty \left(\int_x^\infty \kappa(t - x)|f(t)|dt \right)^p dx \right)^{\frac{1}{p}} \le C\mathfrak{K}(r)\|f\|_{p,\mathfrak{K}}, \end{aligned}$$

where we have applied the condition *(dHC)*$_p$ in the last inequality.

(2) Let $I_2 := \left(\int_{2r}^{\infty} \mathfrak{K}^p(s)\left(\int_{s-r}^{s}\kappa(t+r-s)|f(t)|dt\right)^p ds\right)^{\frac{1}{p}}$. We use similar ideas as in (1), in particular that κ satisfies *(dHC)*$_p$ and *(DC)* to obtain

$$\begin{aligned} I_2 &= \left(\int_r^{\infty}\left[\int_0^{x+r}\kappa(u)du\right]^p\left(\int_x^{x+r}\kappa(t-x)|f(t)|dt\right)^p dx\right)^{\frac{1}{p}} \\ &\leq C\left(\int_r^{\infty}\left[\int_0^{x}\kappa(y)dy\right]^p\left(\int_0^{r}\kappa(u)|f(u+x)|du\right)^p dx\right)^{\frac{1}{p}} \\ &\leq C\int_0^r\left(\int_r^{\infty}\left[\int_0^{x}\kappa(y)dy\right]^p\kappa^p(u)|f(u+x)|^p dx\right)^{\frac{1}{p}} du \\ &\leq C\int_0^r\kappa(u)\left(\int_{r+u}^{\infty}\left[\int_0^{s}\kappa(y)dy\right]^p|f(s)|^p ds\right)^{\frac{1}{p}} du \leq C\mathfrak{K}(r)\|f\|_{p,\mathfrak{K}}, \end{aligned}$$

and we conclude the result. □

The decreasing integral condition

Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a nonnegative function. We say that κ satisfies the decreasing integral condition *(DIC)* if $\int_0^{\varepsilon}\kappa(y)dy > 0$ for all $\varepsilon > 0$ and there exists $C_\kappa > 0$ such that

$$(DIC) \qquad \frac{\int_0^{u+r}\kappa(y)dy}{\int_0^{u+s}\kappa(y)dy} \leq C_\kappa\frac{\int_0^{r}\kappa(y)dy}{\int_0^{s}\kappa(y)dy}, \qquad 0\leq s\leq r, \quad u\geq 0.$$

The *(DIC)* condition is a technical tool which often appears in real analysis and measure theory, see for example level intervals and level functions in [OK, Appendix].

Proposition 5.1.21. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a nonnegative function which satisfies (DC). Then κ satisfies (DIC).*

Proof. Take $r \geq s \geq 0$ and $u > 0$. In the case that $r \leq u$, we have that

$$\frac{\int_0^{u+r}\kappa(y)dy}{\int_0^{u+s}\kappa(y)dy} \leq \frac{\int_0^{2u}\kappa(y)dy}{\int_0^{u}\kappa(y)dy} \leq D_\kappa \leq D_\kappa\frac{\int_0^{r}\kappa(y)dy}{\int_0^{s}\kappa(y)dy}$$

and in the case that $r \geq u$,

$$\frac{\int_0^{u+r}\kappa(y)dy}{\int_0^{u+s}\kappa(y)dy} \leq \frac{\int_0^{2r}\kappa(y)dy}{\int_0^{s}\kappa(y)dy} \leq D_\kappa\frac{\int_0^{r}\kappa(y)dy}{\int_0^{s}\kappa(y)dy},$$

and we conclude the proof. □

Remarks 5.1.22. Note that *(DC)* is not equivalent to *(DIC)*: functions $e_{-\lambda}$ for $\lambda > 0$ satisfy *(DIC)* but not *(DC)*.

The characteristic function $\chi_{(0,1)}$ and $\mathfrak{r}_\nu e_\lambda$ (with $\nu \geq 0$ and $\lambda > 0$) satisfy *(DIC)* (in fact verify *(DC)*, see Lemma 5.1.19). For the characteristic function $\chi_{(1,\infty)}$, *(DIC)* does not hold.

Lemma 5.1.23. *Let $p, q \geq 1$ and $\kappa \in L^1_{loc}(\mathbb{R}^+)$ a positive function. If κ satisfies the (DIC) condition, then*

$$\int_0^r \mathfrak{K}^q(s) \left(\int_0^\infty \left(\frac{\kappa(u+r)}{\mathfrak{K}(u+s)} \right)^p du \right)^{\frac{q}{p}} ds \leq C r \mathfrak{K}^q(r) \left(\int_0^\infty \left(\frac{\kappa(u+r)}{\mathfrak{K}(u+r)} \right)^p du \right)^{\frac{q}{p}},$$

for $r \geq 0$, where $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$.

Proof. We apply the definition of *(DIC)* to get

$$\begin{aligned}
&\int_0^r \mathfrak{K}^q(s) \left(\int_0^\infty \left(\frac{\kappa(u+r)}{\mathfrak{K}(u+s)} \right)^p du \right)^{\frac{q}{p}} ds \\
&\qquad \leq C \int_0^r \mathfrak{K}^q(s) \left(\int_0^\infty \frac{\left(\int_0^r \kappa(y)dy\right)^p}{\left(\int_0^s \kappa(y)dy\right)^p} \left(\frac{\kappa(u+r)}{\int_0^{u+r} \kappa(y)dy} \right)^p du \right)^{\frac{q}{p}} ds \\
&\qquad = C \mathfrak{K}^q(r) \left(\int_0^\infty \left(\frac{\kappa(u+r)}{\int_0^{u+r} \kappa(y)dy} \right)^p du \right)^{\frac{q}{p}} \left(\int_0^r ds \right)
\end{aligned}$$

for $r \geq 0$. $\square$

The Ariño-Muckenhoupt condition

Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a nonnegative function and $1 < p < \infty$ with q its conjugate exponent. We say that κ satisfies *$(AMC)_p$* (Ariño-Muckenhoupt condition) if $\int_0^\varepsilon \kappa(y)dy > 0$ for all $\varepsilon > 0$ and there exists $C_\kappa > 0$ such that

$$(AMC)_p \qquad r^{\frac{1}{q}} \left(\int_r^\infty \left(\frac{\kappa(u)}{\int_0^u \kappa(y)dy} \right)^p du \right)^{\frac{1}{p}} \leq C_\kappa, \qquad r \geq 0.$$

The well-known Ariño-Muckenhoupt theorem states that the weighted Hardy inequality

$$\left(\int_0^\infty \left| \int_0^x f(t)dt \right|^m u(x)dx \right)^{\frac{1}{m}} \leq C \left(\int_0^\infty |f(x)|^p v(x)dx \right)^{\frac{1}{p}}$$

holds for $1 \leq p \leq m \leq \infty$ if and only if

$$\sup_{r>0} \left(\int_r^\infty u(x)dx \right)^{\frac{1}{m}} \left(\int_0^r v(x)^{1-q}dx \right)^{\frac{1}{q}} < \infty, \tag{5.10}$$

see for example [KuMP, p. 44]. Note that *$(AMC)_p$* is, in fact, a particular case of (5.10) for $m = p$, $u(x) = \left(\frac{\kappa(x)}{\mathfrak{K}(x)}\right)^p$ and $v(x) = 1$ for $x \geq 0$. Then *$(AMC)_p$* holds if and only if

$$\|\chi_{(0,\infty)} * f\|_{p,\frac{\kappa}{\mathfrak{K}}} \leq C\|f\|_p, \qquad f \in L_p(\mathbb{R}^+),$$

for $1 \leq p < \infty$ and $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$.

Remark 5.1.24. We just need Hölder's inequality to proof that, if κ satisfies *(AMC)*$_{p_1}$ and *(AMC)*$_{p_2}$, with $1 \leq p_1 \leq p_2 < \infty$, then κ satisfies *(AMC)*$_p$ for all $p \in [p_1, p_2]$.

The characteristic function $\chi_{(0,1)}$ satisfies *(AMC)*$_p$ for all $p > 1$, nevertheless $\chi_{(1,\infty)}$ does not satisfy *(AMC)*$_p$ for any $p \geq 1$. The functions $\mathfrak{r}_\nu e_\lambda$ for $\nu > 0$ and $\lambda \geq 0$ satisfy *(AMC)*$_p$ for $p > 1$ as follows:

$$\begin{aligned} r^{\frac{1}{q}} \left(\int_r^\infty \left(\frac{u^{\nu-1} e^{-\lambda u}}{\int_0^u s^{\nu-1} e^{-\lambda s} ds} \right)^p du \right)^{\frac{1}{p}} &= r^{\frac{1}{q}} \left(\int_r^\infty \frac{u^{p(\nu-1)}}{\left(\int_0^u s^{\nu-1} e^{\lambda(u-s)} ds\right)^p} du \right)^{\frac{1}{p}} \\ &\leq \nu r^{\frac{1}{q}} \left(\int_r^\infty \frac{du}{u^p} \right)^{\frac{1}{p}} = \frac{\nu}{(p-1)^{\frac{1}{p}}}, \end{aligned}$$

for $r > 0$. However, $e_{-\lambda}$ does not satisfy *(AMC)*$_p$ for any $p \geq 1$.

In the next lemma, we prove that there does not exist a non-negative function $\kappa \in L^1_{loc}(\mathbb{R}^+)$ such that satisfies *(AMC)*$_1$.

Lemma 5.1.25. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a nonnegative function such that $\int_0^\varepsilon \kappa(r) dr > 0$ for all $\varepsilon > 0$. Then*

$$\int_0^\infty \frac{\kappa(u)}{\int_0^u \kappa(r) dr} du = \infty.$$

Proof. Suppose that $\int_0^\infty \frac{\kappa(u)}{\int_0^u \kappa(r)dr} du < \infty$. Take $1 > \varepsilon' > 0$ such that $\int_0^{\varepsilon'} \kappa(r) dr > 0$. Then

$$\infty > \int_0^\infty \frac{\kappa(u)}{\int_0^u \kappa(r) dr} du \geq \int_0^\varepsilon \frac{\kappa(u)}{\int_0^u \kappa(r) dr} du \geq \int_0^\varepsilon \frac{\kappa(u)}{\int_0^\varepsilon \kappa(r) dr} du = 1$$

for any $0 < \varepsilon < \varepsilon'$. By the dominated convergence theorem, we conclude that

$$1 \leq \lim_{\varepsilon \to 0^+} \int_0^\varepsilon \frac{\kappa(u)}{\int_0^u \kappa(r) dr} du = 0$$

and we conclude the proof of the lemma. □

Corollary 5.1.26. *Let $p > 1$, with q such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a positive function. If κ satisfies (DIC) and $(AMC)_p$ then*

$$\int_0^r \left(\int_0^s \kappa(y) dy \right)^q \left(\int_0^\infty \left(\frac{\kappa(u+r)}{\int_0^{u+s} \kappa(y) dy} \right)^p du \right)^{\frac{q}{p}} ds \leq C \left(\int_0^r \kappa(y) dy \right)^q$$

for $r \geq 0$.

Proof. We apply Lemma 5.1.23 and the *(AMC)*$_p$ condition to get the result. □

Theorem 5.1.27. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ a nonnegative function such that satisfies (DIC) and (AMC)$_q$ for some $q>1$ (with p its conjugate exponent) and $\mathfrak{K} = \chi_{(0,\infty)} * \kappa$. Then there exist a constant $C>0$, such that*

$$\left(\int_0^\infty \mathfrak{K}^p(s)\left(\int_0^s |g(r)|\int_s^\infty \kappa(t+r-s)|f(t)|dtdr\right)^p ds\right)^{\frac{1}{p}} \le C\|f\|_{p,\mathfrak{K}}\|g\|_{1,\mathfrak{K}} \tag{5.11}$$

for $f \in L_p(\mathfrak{K}^p)$, $g \in L_1(\mathfrak{K})$.

Proof. Let

$$J := \left(\int_0^\infty \mathfrak{K}^p(s)\left(\int_0^s |g(r)|\int_s^\infty \kappa(t+r-s)|f(t)|dtdr\right)^p ds\right)^{\frac{1}{p}}$$

for $f \in L_p(\mathfrak{K}^p)$, and $g \in L_1(\mathfrak{K})$. Then

$$\begin{aligned} J &\le \left(\int_0^\infty \left(\int_s^\infty |g(r)|\left(\int_0^s \kappa(y)dy\right)\int_s^\infty \kappa(t+r-s)|f(t)|dtdr\right)^p ds\right)^{\frac{1}{p}} \\ &\le \int_0^\infty |g(r)|\left(\int_0^r \left(\int_0^s \kappa(y)dy\right)^p \left(\int_s^\infty \kappa(t+r-s)|f(t)|dt\right)^p ds\right)^{\frac{1}{p}} dr. \end{aligned}$$

Now, we apply Hölder's inequality to obtain

$$\int_s^\infty \frac{\kappa(t+r-s)}{\int_0^t \kappa(y)dy}|f(t)|\left(\int_0^t \kappa(y)dy\right)dt \le \left(\int_s^\infty \left(\frac{\kappa(t+r-s)}{\int_0^t \kappa(u)du}\right)^q dt\right)^{\frac{1}{q}} \|f\|_{p,\mathfrak{K}},$$

and then

$$J \le \|f\|_{p,\mathfrak{K}}\int_0^\infty |g(r)|\left(\int_0^r \left(\int_0^s \kappa(y)dy\right)^p \left(\int_0^\infty \left(\frac{\kappa(u+r)}{\int_0^{u+s}\kappa(y)dy}\right)^q du\right)^{\frac{p}{q}} ds\right)^{\frac{1}{p}} dr.$$

Now we apply Corollary 5.1.26 to conclude that

$$J \le C\|f\|_{p,\mathfrak{K}}\int_0^\infty |g(r)|\left(\int_0^r \kappa(y)dy\right)dr \le C\|f\|_{p,\mathfrak{K}}\|g\|_{1,\mathfrak{K}}.$$

□

Remark 5.1.28. For $p=1$, inequality (5.11) holds with κ satisfying *(DC)* without any additional condition (i.e., *(AMC)*$_\infty$): we just apply the Fubini theorem.

5.2 Sharp extensions for convoluted solutions of abstract Cauchy problems

Let A be a closed linear operator on a Banach space X. Let $\kappa : (0,\tau) \to \mathbb{C}$ be a locally integrable function, and $x \in X$. The Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \mathfrak{K}(t)x & 0 < t < \tau, \\ v(0) = 0, \end{cases} \tag{5.12}$$

is called $\mathfrak{K}$-convoluted Cauchy problem where $\mathfrak{K}(t) = \int_0^t \kappa(s)xds$ for $0 < t < \tau$. If there exists a solution of the abstract initial value problem $u'(t) = Au(t)$ for $0 < t < \tau$, $u(0) = x$ then, as usual for a nonhomogeneous equation, we have $v = u * \mathfrak{K}$ ($*$ is the usual convolution in $\mathbb{R}^+$). Local κ-convoluted semigroups are defined using a version of Duhamel's formula and were introduced in [C, CL]. This class of semigroups includes C_0-semigroups and integrated semigroups as particular examples, see a complete treatment in [MF, Section 1.3.1], [Ko, Chapter 2] and other details in [KLM, KP]. The concept of regularized semigroups is covered by taking $\mathfrak{K}(t) \equiv C$, $0 \le t < \tau$, where C is a bounded and injective operator on X. In this case as in (6), we adopt the convention that $\kappa(t) = C\delta_0$ (resp. $\kappa(t) = C\delta_0$) ($\delta_0 I$ is the Dirac measure concentrated at 0) and then $\mathfrak{K}(t) = CH(t)$ (resp. $\mathfrak{K}(t) = H(t)I$, where H is the Heaviside function. Note that here, κ and $\mathfrak{K}$ are operator-valued.

Contrary to what happens in the case of equation (6), if the function $\kappa : (0,\infty) \to \mathbb{C}$ is locally integrable and for every $x \in X$ there exists a unique solution $u \in C^1([0,\tau), X) \cap C([0,\tau), D(A))$ for (5.12), it is generally not the case that these solutions can be extended to $[0,\infty)$, nor that exponential boundedness is achieved in case one can extend the solutions. In this case, we say that A is the generator of a local κ-convoluted semigroup. However, there is an underlying algebraic structure of κ-convoluted semigroups which leads to the following extension property: the solution of the κ-convoluted Cauchy problem on $[0,\tau)$ is used to express the solution of the $\kappa * \kappa$-convoluted problem on $[0,2\tau)$, see [CL, Section 2] and [Ko, Theorem 2.1.1.9]. Stated otherwise, when (5.12) is well posed on $[0,\tau)$, the equation in which we replace $\kappa(\cdot)$ with $(\kappa * \kappa)(\cdot)$ is well posed on $[0,2\tau)$. Our result (Theorem 5.2.17) provide a sharpening of this extension property.

The special case of $\kappa = \mathfrak{r}_\nu$ with $\nu > 0$ defines the ν-times integrated semigroup. Originally they were the first example of convoluted semigroups. An extension formula for n-times integrated semigroups (for $n \in \mathbb{N}$) was given in [AEK, Section IV, (4.2)] and for ν-times integrated semigroup in [Mi4, Formula (5)] with $\nu > 0$. Extensions of local ν-times integrated C-semigroups were given in [LS, Theorem 6.1] and automatic extension of local regularized semigroups appears in [WG, Section 2].

The main objective of this section is to illustrate the algebraic structure of local κ-convoluted semigroups. In [KLM, Section 5], the authors consider global exponentially bounded convoluted semigroups and algebra homomorphisms defined via these classes of semigroups; in fact both concepts are equivalent, see [KLM, Theorem 5.7].

In the context of local convoluted semigroups as well as global non-exponentially bounded convoluted semigroups, this point of view is not so evident. This is due to the fact that the Laplace transform is an essential tool in the global exponential case. First we need some technical identities which involve convolution products. We introduce a new test function space, $\mathcal{D}_{\kappa^{*\infty}}$ (in Definition 5.2.9) which will play a fundamental role, see Theorem 5.2.22 and SUBSECTION 5.16. Then we give one of the main results of this chapter. We derive a sharp extension theorem for local convoluted semigroups (Theorem 5.2.17). We use the extension formula to define algebra homomorphism from the test function space $\mathcal{D}_{\kappa^{*\infty}}$ via local convoluted semigroup (Theorem 5.2.22). To end the section, we apply our results to four concrete operators which generate (local and global) convoluted semigroups. Again we emphasize that in the global case and under the exponential boundedness assumption, the Laplace transform is used as a crucial tool. This is no longer the case when one consider the local case or the global one without the assumption of exponential boundedness. In the case we consider, algebras concerned are no longer Banach algebras but only locally convex algebras.

Historically distribution semigroups were introduced by J.L. Lions in the seminal paper [Li] in the early sixties in the exponential case with the Laplace transform of vector-valued distributions as an important tool. The paper [Ch] by J. Chazarain presents an extension to the non exponential case and goes further to introduce the ultradistributional framework (see also the monograph [LM]). This class of vector-valued distribution (with a suitable algebraic structure) gives an equivalent approach to local integrated semigroups as was proven in [AEK, Theorem 7.2]. For local convoluted semigroups, we present a similar approach in Subsection 5.2.6, where we introduce κ-distribution semigroups and we present their connections with local convoluted semigroups. The interest in the local case stems from the fact that for the general classes of generalized semigroups that have been introduced following Lions' paper, by using the local approach, one is able to obtain a Banach space valued formulation that captures almost all the situations involved. The monographs [Ko], [MF] and the references cited therein contain more information on distribution as well as ultradistribution semigroups. They also explore the ways in which they relate to local convoluted semigroups.

A similar and independent approach may be followed in the abstract Cauchy problem of second order or wave problem. In this case we need to consider local convoluted cosine functions and distribution cosine function (and the corresponding algebra homomorphism for the cosine convolution product) see more details in [MP].

5.2.1 Several equalities for convolution products

Considering the convolution product $*$, we write κ^{*2} instead of $\kappa*\kappa$ and then by induction $\kappa^{*n} = \kappa * \left(\kappa^{*(n-1)}\right)$ for $n > 2$ is the n-fold convolution power of κ. The convolution product is associative and commutative. If we also consider the dual convolution product $\circ$, note that

$$\max\{t|\, t \in \operatorname{supp}(f \circ g)\} \leq \max\{t|\, t \in \operatorname{supp}(g)\}, \qquad f, g \in L_1(\mathbb{R}^+).$$

We denote by χ the constant function equals to 1, i.e., $\chi(t) = 1$ for $t \in \mathbb{R}^+$. This corresponds to the Heaviside function. Consider functions $\mathfrak{r}_\nu(t)$, for $\nu > 0$ and $t \geq 0$. It will be convenient to set $\mathfrak{r}_0 = \delta_0$, the Dirac measure concentrated at the origin. Observe that the following semigroup property holds: $\mathfrak{r}_\nu * \mathfrak{r}_\mu = \mathfrak{r}_{\nu+\mu}, \nu, \mu \geq 0$. The following lemma will be used for the proof of the main result, Theorem 5.2.17.

Lemma 5.2.1. *Take* $0 \leq \tau \leq t$ *and* $f, g \in L^1_{loc}(\mathbb{R}^+)$. *Then*

$$\int_0^{t-\tau} f(t-s)\,(\chi * g)\,(s)ds \;+\; \int_0^{\tau} g(t-s)\,(\chi * f)\,(s)ds$$
$$= \; (g * (\chi * f))\,(t) - (\chi * g)\,(t-\tau)\,(\chi * f)\,(\tau).$$

Proof. Observe that $\dfrac{d}{ds}\displaystyle\int_s^t f(t-u)du = -f(t-s)$ and by simple change of variable we have:

$$\int_{t-\tau}^t f(t-u)du = \int_0^\tau f(s)ds \quad \text{and} \quad \int_s^t f(t-u)du = \int_0^{t-s} f(u)du.$$

We integrate by parts in the following integral to obtain,

$$\begin{aligned}
&\int_0^{t-\tau} f(t-s)\int_0^s g(u)duds\\
&\qquad = -\int_{t-\tau}^t f(t-s)ds\int_0^{t-\tau} g(u)du + \int_0^{t-\tau} g(s)\int_s^t f(t-u)duds\\
&\qquad = -\,(\chi * g)\,(t-\tau)\,(\chi * f)\,(\tau) + \int_0^{t-\tau} g(s)\int_0^{t-s} f(x)dxds,
\end{aligned}$$

for $0 \leq \tau \leq t$. Note that

$$\begin{aligned}
\int_0^{t-\tau} g(s)\int_0^{t-s} f(x)dxds &= \int_0^t g(s)\int_0^{t-s} f(x)dxds - \int_{t-\tau}^t g(s)\int_0^{t-s} f(x)dxds\\
&= (g * (\chi * f))\,(t) - \int_0^\tau g(t-u)\int_0^u f(x)dxdu\\
&= (g * (\chi * f))\,(t) - \int_0^\tau g(t-u)\,(\chi * f)\,(u)du,
\end{aligned}$$

and this concludes the proof. $\square$

Taking $f = \mathfrak{r}_\nu$ and $g = \mathfrak{r}_\mu$ with $\nu, \mu > 0$ in Lemma 5.2.1, we get the equality

$$\int_0^{t-\tau} \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}\frac{s^\mu}{\Gamma(\mu+1)}ds + \int_0^\tau \frac{(t-s)^{\mu-1}}{\Gamma(\mu)}\frac{s^\nu}{\Gamma(\nu+1)}ds$$
$$= \frac{t^{\nu+\mu}}{\Gamma(\nu+\mu+1)} - \frac{(t-\tau)^\mu}{\Gamma(\mu+1)}\frac{\tau^\nu}{\Gamma(\nu+1)}$$

for $0 \leq \tau \leq t$.

If we now set $f = g$ in Lemma 5.2.1, we obtain:

Corollary 5.2.2. *Take $0 \le \tau \le t$ and $f \in L^1_{loc}(\mathbb{R}^+)$. Then*

$$\left(\int_0^{t-\tau} + \int_0^{\tau}\right) f(t-s)\,(\chi * f)\,(s)ds = (f*(\chi*f))\,(t) - (\chi*f)\,(t-\tau)\,(\chi*f)\,(\tau). \tag{5.13}$$

Further specializing to $f = \mathfrak{r}_\nu$ for $\nu > 0$, yields the identity:

$$\left(\int_0^{t-\tau} + \int_0^{\tau}\right) \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} \frac{s^\nu}{\Gamma(\nu+1)} ds = \frac{t^{2\nu}}{\Gamma(2\nu+1)} - \frac{(t-\tau)^\nu}{\Gamma(\nu+1)} \frac{\tau^\nu}{\Gamma(\nu+1)}$$

for $0 \le \tau \le t$.

As a consequence of the last corollary, we obtain another proof of the following result given in [Ko, Lemma 2.1.12] for continuous functions and in [KuS, Lemma 3.1] for $f = \mathfrak{r}_\nu$ and $\nu > 0$.

Corollary 5.2.3. *For $f \in L^1_{loc}(\mathbb{R}^+)$ and $s, u \ge 0$ we have*

$$\left(\int_0^{s+u} - \int_0^{s} - \int_0^{u}\right) f(u+s-r)f(r)dr = 0;$$

in particular for $\nu > 0$ and $s, u \ge 0$, we get that

$$\left(\int_0^{s+u} - \int_0^{s} - \int_0^{u}\right) (u+s-r)^{\nu-1} r^{\nu-1} dr = 0;$$

Proof. By change of variable, we write the identity (5.13) as

$$(f*(\chi*f))\,(t) - (\chi*f)\,(t-\tau)\,(\chi*f)\,(\tau) = \left(\int_\tau^{t} + \int_{t-\tau}^{t}\right) f(x)(\chi*f)(t-x)dx$$

for $0 \le \tau \le t$.

Now we observe that

$$\frac{d}{dt}(f*(\chi*f))(t) = (f*f)(t).$$

Similarly, we have

$$\frac{d}{dt}((\chi*f)(t-\tau)(\chi*f)(\tau)) = f(t-\tau)(\chi*f)(\tau),$$

$$\begin{aligned} \frac{d}{dt}\int_t^{\tau} f(u)(\chi*f)(t-u)du &= -f(t)(\chi*f)(0) - \int_\tau^{t} f(u)f(t-u)du \\ &= \int_t^{\tau} f(u)f(t-u)du, \end{aligned}$$

and

$$\frac{d}{dt}\int_{t-\tau}^{t} f(u)(\chi * f)(t-u)du$$
$$= f(t)(\chi * f)(0) + \int_0^t f(u)f(t-u)du - f(t-\tau)(\chi * f)(\tau) - \int_0^{t-\tau} f(u)f(t-u)du$$
$$= \int_0^t f(u)f(t-u)du - f(t-\tau)(\chi * f)(\tau) - \int_0^{t-\tau} f(u)f(t-u)du.$$

Differentiating with respect to the variable t and using the above, we have:

$$\begin{aligned}(f * f)(t) &= \int_\tau^t f(x)f(t-x)dx + \int_{t-\tau}^t f(x)f(t-x)dx \\ &= \int_0^{t-\tau} f(t-s)f(s)ds + \int_0^\tau f(t-s)f(s)ds\end{aligned}$$

for $0 \le \tau \le t$. Now take $t = s + u$, and $\tau = s$ and we conclude the proof. □

Take $\kappa \in L^1_{loc}([0,\tau))$, and we define $(\kappa_t)_{t\in[0,\tau)} \subset L^1_{loc}([0,\tau))$ by

$$\kappa_t(s) := \kappa(t-s)\chi_{[0,t]}(s), \qquad s \in [0,\tau). \tag{5.14}$$

A similar result was considered in [KLM, Proposition 2.2] for functions belonging to $L^1_{loc}(\mathbb{R}^+)$. Here we present a direct proof for $L^1_{loc}([0,\tau))$.

Theorem 5.2.4. *Take $\kappa \in L^1_{loc}([0,\tau))$ and $(\kappa_t)_{t\in[0,\tau)}$ defined by (5.14). Then*

$$\kappa_t * \kappa_s(x) = \int_t^{t+s} \kappa(t+s-r)\kappa_r(x)dr - \int_0^s \kappa(t+s-r)\kappa_r(x)dr, \qquad 0 \le x < \tau,$$

for $0 \le s, t \le t+s < \tau$.

Proof. We consider (without lost of generalization) that $0 \le s \le t$. First we consider $0 \le x \le s$. Then

$$\kappa_t * \kappa_s(x) = \int_0^x \kappa(t-(x-y))\kappa(s-y)dy, \quad 0 \le x < \tau,$$

and

$$\begin{aligned}\left(\int_t^{t+s} - \int_0^s\right)\kappa(t+s-r)\kappa_r(x)dr &= \left(\int_t^{t+s} - \int_x^s\right)\kappa(t+s-r)\kappa(r-x)dr \\ = \int_0^s \kappa(u)\kappa(t+s-u-x)du &- \int_0^{s-x} \kappa(t+s-x-y)\kappa(y)dy \\ = \int_{s-x}^s \kappa(u)\kappa(t+s-u-x)du &= \int_0^x \kappa(s-y)\kappa(t-x+y)dy\end{aligned}$$

where we have changed variables in several equalities. The other cases $s \le x \le t$, $t \le x \le t+s$ and $t+s \le x < \tau$ are made following similar ideas. In particular we remark that $\kappa_t * \kappa_s(x) = 0$ for $t+s \le x < \tau$. □

Remark 5.2.5. By Proposition 5.2.16 and Theorem 5.2.4, we may conclude that $(\kappa_t)_{t\in[0,\tau)}$ is a local κ-convoluted semigroup in $L^1_{loc}([0,\tau))$. In fact, note that $\kappa_t = \delta_t * \kappa$ where $(\delta_t)_{t\geq 0}$ is the Dirac measure concentrated at t. In this sense, $(\kappa_t)_{t\in[0,\tau)}$ is the canonical local κ-convoluted semigroup.

5.2.2 The Laplace transform and κ-test function spaces

Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$. We write by $\mathcal{L}\kappa$ the usual Laplace transform of κ, given by

$$\mathcal{L}\kappa(z) = \lim_{N\to\infty}\int_0^N e^{-zt}\kappa(t)dt,$$

in the case that there exists for some $z\in\mathbb{C}$; $\operatorname{abs}(\kappa)$ is defined by $\operatorname{abs}(\kappa) := \inf\{\operatorname{Re} z;\ \text{exist } \mathcal{L}\kappa(z)\}$, see [ABHN, Section 1.4]. In the case that $|\kappa(t)|\leq Me^{\omega t}$ for a.e. $t\geq 0$ and $M,\omega>0$, we have that

$$\mathcal{L}\kappa(z) = \int_0^\infty e^{-zt}\kappa(t)dt, \qquad \operatorname{Re} z > \omega.$$

For $z\in\mathbb{C}$, recall that we write $e_z(t) = e^{-zt}$, $t\geq 0$.

Lemma 5.2.6. *Take $\kappa \in L^1_{loc}(\mathbb{R}^+)$ such that $|\kappa(t)|\leq Me^{Bt}$ for a.e. $t\geq 0$ and $M,B>0$. Then*

$$\kappa\circ e_z = \mathcal{L}\kappa(z)e_z, \qquad \operatorname{Re} z > B.$$

In particular, if $\mathcal{L}\kappa(z)=0$ for some $\operatorname{Re} z > B$, then $\kappa\circ e_z = 0$.

Proof. Take $z\in\mathbb{C}$ with $\operatorname{Re} z > B$ and

$$\kappa\circ e_z(t) = \int_t^\infty \kappa(s-t)e^{-zs}ds = e^{-zt}\int_0^\infty \kappa(u)e^{-zu}du = \mathcal{L}\kappa(z)e_z(t)$$

for $t>0$. □

Just to simplify the notation, consider $\mathcal{D}$ as the space of $C^{(\infty)}$ functions with compact support on $\mathbb{R}$ (that is, $\mathcal{D} := C_c^{(\infty)}(\mathbb{R})$) and $\mathcal{D}_0$ as the subspace of $\mathcal{C}^{(\infty)}$ functions with compact support on $[0,\infty)$ (that is, $\mathcal{D}_0 := C_c^{(\infty)}[0,\infty)$). $\mathcal{D}_0\subset\mathcal{D}$. The space $\mathcal{D}$ will be equipped with the Schwartz topology which turns it into a complete topological vector space. We denote the topology by $\mathcal{T}$. In particular, sequential convergence in $\mathcal{D}$ is described by: let $(\phi_n)_{n\geq 1}\subset\mathcal{D}$, $\phi\in\mathcal{D}$, then $\phi_n\to\phi$ if and only if

(1) there exists a compact subset $K\subset\mathbb{R}$ such that $\operatorname{supp}(\phi_n), \operatorname{supp}(\phi)\subset K$.

(2) for any $j\geq 0$, $\phi_n^{(j)}\to\phi^{(j)}$ uniformly on compact sets.

Note that $\mathcal{D}_0$ is a closed subspace of $\mathcal{D}$ and then $(\mathcal{D}_0,\mathcal{T})$ is a complete topological space (we keep the same notation for the topology $\mathcal{T}$ and its restriction to the subspace $\mathcal{D}_0$).

We denote by $\mathcal{D}_+$ the set of functions defined by $\phi_+ : [0,\infty)\to\mathbb{C}$, given by $\phi_+(t) := \phi(t)$ for $t\geq 0$ and $\phi\in\mathcal{D}$ and define $\mathcal{K}:\mathcal{D}\to\mathcal{D}_+$ by $\mathcal{K}(\phi)=\phi_+$ for $\phi\in\mathcal{D}$. Due to the extension theorem of R. T. Seeley [Se], there exists a linear continuous operator $\Lambda : \mathcal{D}_+ \to \mathcal{D}$, such that $\mathcal{K}\Lambda = I_{\mathcal{D}_+}$; in particular if ψ is a $\mathcal{C}^{(\infty)}$ function on $[0,\infty)$ and compact support then $\psi \in \mathcal{D}_+$. The space $\mathcal{D}_+$ is also a complete topological vector space equipped with the $\mathcal{T}$-topology of uniform convergence on bounded subsets.

We define the operator $T'_\kappa : \mathcal{D} \to \mathcal{D}$ by $f \mapsto T'_\kappa(f) := \kappa \circ f$, that is,

$$T'_\kappa(f)(t) = \int_t^\infty \kappa(s-t)f(s)ds, \qquad t \geq 0.$$

We shall also use the same notation for the restriction to $\mathcal{D}_+$, $T'_\kappa : \mathcal{D}_+ \to \mathcal{D}_+$; however $T'_\kappa : \mathcal{D}_0 \not\to \mathcal{D}_0$. Note that $T'_\kappa(f_u) = (T'_\kappa(f))_u$, where $f_u(t) = f(u+t)$ for $u, t \geq 0$ and $f \in \mathcal{D}_+$.

In the case that $0 \in \mathrm{supp}(\kappa)$, we have that $T'_\kappa : \mathcal{D}_+ \to \mathcal{D}_+$ is an injective, linear and continuous homomorphism such that

$$T'_\kappa(f \circ g) = f \circ T'_\kappa(g), \qquad f, g \in \mathcal{D}_+,$$

see [KLM, Theorem 2.5]. Then, we define the space $\mathcal{D}_\kappa$ by $\mathcal{D}_\kappa := T'_\kappa(\mathcal{D}_+) \subset \mathcal{D}_+$ and the right inverse map of T'_κ, i.e., $W_\kappa : \mathcal{D}_\kappa \to \mathcal{D}_+$ by

$$f(t) = T'_\kappa(W_\kappa(f))(t) = \int_t^\infty \kappa(s-t)W_\kappa f(s)ds, \qquad f \in \mathcal{D}_\kappa, \quad t \geq 0,$$

see [KLM, Definition 2.7]. Note that the operator $W_\kappa : \mathcal{D}_+ \to \mathcal{D}_+$ is a closed operator ($D(W_\kappa) = \mathcal{D}_\kappa$), but we cannot apply the open mapping theorem to conclude that it is continuous.

It is clear that the subspace $\mathcal{D}_\kappa$ is also a topological algebra: take $f, g \in \mathcal{D}_\kappa$, then $f * g \in \mathcal{D}_\kappa$ ([KLM, Theorem 2.10]) and the map $(f,g) \to f * g$ is continuous in $\mathcal{D}_\kappa$. Moreover $W_\kappa(\kappa \circ f) = f$ for $f \in \mathcal{D}_+$ and $f_u \in \mathcal{D}_\kappa$, with

$$W_\kappa(f_u) = (W_\kappa(f))_u, \qquad f \in \mathcal{D}_\kappa, \quad u \geq 0. \tag{5.15}$$

We have the following property to the effect that W_κ does not increase the support.

Lemma 5.2.7. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be such that $0 \in supp(\kappa)$ and let $a > 0$. Then $supp(f) \subset [0,a]$ if and only if $supp(W_\kappa f) \subset [0,a]$ for $f \in \mathcal{D}_\kappa$.*

Proof. Take $f \in \mathcal{D}_\kappa$ such that $\mathrm{supp}(f) \subset [0,a]$. Then

$$0 = f(t) = \kappa \circ W_\kappa f(t) = \int_t^\infty \kappa(s-t)W_\kappa f(s)ds, \qquad t \geq a,$$

We write $t = a + r$ for $t \geq a$ and $r \geq 0$,

$$0 = \int_t^\infty \kappa(s-t)W_\kappa f(s)ds = \int_0^\infty \kappa(x)W_\kappa f(x+a+r)dx = \int_r^\infty \kappa(x-r)(W_\kappa f)_a(x)dx,$$

where $(W_\kappa f)_a(x) := (W_\kappa f)(x+a)$ for $x > 0$. We apply the Titchmarsh-Foiaş Theorem [KLM, Theorem 2.4] to conclude that $(W_\kappa f)_a(x) = 0$ for $x > 0$, i.e., $\mathrm{supp}(W_\kappa f) \subset [0, a]$. Conversely, suppose that $\mathrm{supp}(W_\kappa f) \subset [0, a]$. It then follows from the representation

$$f(t) = \int_t^\infty \kappa(s-t) W_\kappa f(s) ds, \qquad t \geq 0,$$

that $\mathrm{supp}(f) \subset [0, a]$. □

Note that in the case that $f \in \mathcal{D}_\kappa$ then $f^{(n)} \in \mathcal{D}_\kappa$ and

$$W_\kappa(f^{(n)}) = (W_\kappa f)^{(n)}, \qquad n \geq 1; \tag{5.16}$$

take $\kappa, l \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in \mathrm{supp}(\kappa) \cap \mathrm{supp}(l)$. Then $0 \in \mathrm{supp}(\kappa * l)$, $\mathcal{D}_{\kappa * l} \subset \mathcal{D}_\kappa \cap \mathcal{D}_l$ and

$$W_\kappa f = l \circ W_{\kappa * l} f, \qquad f \in \mathcal{D}_{\kappa * l}. \tag{5.17}$$

see [KLM, Lemma 2.8]. A consequence of (5.17) is the following lemma.

Lemma 5.2.8. *Take $\kappa \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in supp(\kappa)$. Then*

(1) *$\mathcal{D}_{\kappa^{*n}} \subset \mathcal{D}_{\kappa^{*m}}$ for $n \geq m \geq 1$.*

(2) *$W_{\kappa^{*m}} f = \kappa^{n-m} \circ W_{\kappa^{*n}} f = W_{\kappa^{*n}}(\kappa^{n-m} \circ f)$ and if $supp(W_{\kappa^{*n}} f) \subset I$ with I an interval in $\mathbb{R}^+$ then $supp(W_{\kappa^{*m}} f) \subset I$ for $f \in \mathcal{D}_{\kappa^{*n}}$ and $n \geq m \geq 1$.*

The next definition gives the test function space which will be used later to obtain new distribution spaces and corresponding distribution semigroups.

Definition 5.2.9. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be such that $0 \in supp(\kappa)$. We denote by $\mathcal{D}_{\kappa^{*\infty}}$ the space defined by*

$$\mathcal{D}_{\kappa^{*\infty}} := \bigcap_{n=1}^\infty \mathcal{D}_{\kappa^{*n}}.$$

It is clear that $\mathcal{D}_{\kappa^{*\infty}}$ is also a topological algebra (equipped with the $\mathcal{T}$-topology) and $\mathcal{D}_{\kappa^{*\infty}} \hookrightarrow \mathcal{D}_{\kappa^{*n}} \hookrightarrow \mathcal{D}_+$. In fact, $\mathcal{D}_{\kappa^{*\infty}}$ is the inverse (or projective) limit of the family $(\mathcal{D}_{\kappa^{*n}})_{n \geq 1}$. By Lemma 5.2.8, $W_{\kappa^{*n}} : \mathcal{D}_{\kappa^{*\infty}} \to \mathcal{D}_{\kappa^{*\infty}}$ and $\kappa^{*n} \circ W_{\kappa^{*n}} f = f$ for $f \in \mathcal{D}_{\kappa^{*\infty}}$ and $n \in \mathbb{N}$. Note that if $f \in \mathcal{D}_{\kappa^{*\infty}}$ then $f_u \in \mathcal{D}_{\kappa^{*\infty}}$ for $u \geq 0$, see formula (5.15).

Examples 5.2.10. In the case that $\mathcal{D}_\kappa = \mathcal{D}_+$, then $\mathcal{D}_{\kappa^{*n}} = \mathcal{D}_+$ for $n \in \mathbb{N}$ and consequently, $\mathcal{D}_{\kappa^{*\infty}} = \mathcal{D}_+$. Take $z \in \mathbb{C}$, $e_z(t) := e^{-zt}$ for $t \geq 0$, then $\mathcal{D}_{\kappa e_z} = \{e_z f \mid f \in \mathcal{D}_\kappa\} = \mathcal{D}_\kappa$ and

$$W_{\kappa e_z} f = e_z W_\kappa(e_{-z} f), \qquad f \in \mathcal{D}_\kappa;$$

in the case that $\mathcal{D}_\kappa = \mathcal{D}_+$, then $\mathcal{D}_{\kappa e_z} = \mathcal{D}_+$, see [KLM, Proposition 2.9].

(1) As we have said many times through the monograph, for $\nu > 0$ and $\kappa = \mathfrak{r}_\nu$; the map $W_{\mathfrak{r}_\nu}$ is the Weyl fractional derivative of order ν, W^ν, and $\mathcal{D}_{\mathfrak{r}_\nu} = \mathcal{D}_{\mathfrak{r}_\nu^{*\infty}} = \mathcal{D}_+$; for

$\nu \in \mathbb{N}$, $W^\nu = (-1)^\nu \frac{d^\nu}{dt^\nu}$, the ν-iterate of usual derivation, see more details for example in [SKM].

(2) Given $\nu > 0$ and $z \in \mathbb{C}$, we have that $\mathcal{D}_{e_z \mathfrak{r}_\nu} = \mathcal{D}_{(e_z \mathfrak{r}_\nu)^{*\infty}} = \mathcal{D}_+$ and

$$W_{e_z \mathfrak{r}_\nu} f = e_z W^\nu (e_{-z} f), \qquad f \in \mathcal{D}_+;$$

for $\nu = 1, 2$ see explicit expressions in [KLM, Section 2].

(3) It is straightforward to check that $T'_{\chi_{(0,1)}}(f)(t) = \int_t^{t+1} f(s)ds$ for $f \in \mathcal{D}_+$, $\mathcal{D}_{\chi_{(0,1)}} = \mathcal{D}_{\chi_{(0,1)}^{*n}} = \mathcal{D}_+$ and

$$W_{\chi_{(0,1)}} f(t) = -\sum_{n=0}^{\infty} f'(t+n), \qquad f \in \mathcal{D}_+, \quad t \geq 0.$$

Now let $f, g \in \mathcal{D}_\kappa$. Then $f * g \in \mathcal{D}_\kappa$ and

$$W_\kappa (f*g)(s) = \int_0^s W_\kappa g(r) \int_{s-r}^s \kappa(t+r-s) W_\kappa f(t) dt dr - \int_s^\infty W_\kappa g(r) \int_s^\infty \kappa(t+r-s) W_\kappa f(t) dt dr, \tag{5.18}$$

see [KLM, Theorem 2.10].

Under some conditions on the function κ, some Banach algebras under the convolution product may be considered as the next theorem shows.

Theorem 5.2.11. *([KLM, Theorems 3.4 and 3.5]) Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ be a function with $0 \in supp(\kappa)$ and $abs(|\kappa|) < \infty$. Then the formula*

$$\|f\|_{\kappa, e_\beta} := \int_0^\infty |W_\kappa f(t)| e^{\beta t} dt, \qquad f \in \mathcal{D}_\kappa,$$

for $\beta > \max\{abs(|\kappa|), 0\}$ defines an algebra norm on $\mathcal{D}_\kappa$ for the convolution product $*$. We denote by $\mathcal{T}^\kappa(e_\beta)$ the Banach space obtained as the completion of $\mathcal{D}_\kappa$ in the norm $\|\cdot\|_{\kappa, e_\beta}$, and then we have $\mathcal{T}^\kappa(e_\beta) \hookrightarrow L_1(\mathbb{R}^+)$.

Note that in three examples below, the space $\mathcal{D}_\kappa = \mathcal{D}_+$. However, as the following lemma shows, there are functions κ such that $\mathcal{D}_\kappa \subsetneq \mathcal{D}_+$ and $\mathcal{D}_{\kappa^{*\infty}} = \{0\}$.

Theorem 5.2.12. *Take $\kappa \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in supp(\kappa)$ and $0 \leq abs(|\kappa|) < \infty$. If $\mathcal{L}\kappa(\lambda_0) = 0$ for some $\operatorname{Re}\lambda_0 > abs(|\kappa|)$ then $\mathcal{D}_\kappa \subsetneq \mathcal{D}_+$ and $\mathcal{D}_{\kappa^{*\infty}} = \{0\}$.*

Proof. We suppose that $\mathcal{D}_\kappa = \mathcal{D}_+$ and $\mathcal{L}\kappa(\lambda_0) = 0$ for some $\operatorname{Re}\lambda_0 > \operatorname{abs}(|\kappa|)$. Take $\beta \in \mathbb{R}$ such that $\operatorname{abs}(|\kappa|) < \beta < \operatorname{Re}\lambda_0$. There exists $f_n \subset \mathcal{D}_+$ such that

$$\int_0^\infty |W_\kappa f_n(t) - e_{-\lambda_0}(t)| e^{\beta t} dt \to 0, \qquad n \to \infty. \tag{5.19}$$

As a consequence of Theorem 5.2.11, we obtain that $f_n \not\to 0$ in $L_1(\mathbb{R}^+)$. On the other hand, by [KLM, Theorem 2.5 (ii)] and (5.19), we get that

$$\int_0^\infty |f_n(t) - \kappa \circ e_{-\lambda_0}(t)| e^{\beta t} dt \to 0, \qquad n \to \infty.$$

By Lemma 5.2.6 $\kappa \circ e_{-\lambda_0} = 0$, and then $f_n \to 0$ in $L_1(\mathbb{R}^+)$. We conclude that $\mathcal{D}_\kappa \subsetneq \mathcal{D}_+$.

Now take $f \in \mathcal{D}_{\kappa^{*\infty}}$. Then there exists a sequence $(g_n) \subset \mathcal{D}_+$, such that $f = \kappa^{*n} * g_n$ for $n \geq 1$. Then $\mathcal{L}f(\lambda_0) = (\mathcal{L}\kappa(\lambda_0))^n \mathcal{L}g_n(\lambda_0) = 0$ for any $n \geq 1$. We conclude that $\mathcal{L}f$ has a zero on λ_0 of order n at least for any $n \geq 1$. We conclude that $f = 0$. $\square$

Example 5.2.13. The following example was presented in [B, Section 5] and appeared later in other references in connection to convoluted semigroups (see [KP, Example 6.1] and [Ko, Example 2.8.1]). Let

$$K(\lambda) := \frac{1}{\lambda^2} \prod_{n=0}^{\infty} \frac{n^2 - \lambda}{n^2 + \lambda}, \qquad \operatorname{Re} \lambda > 0.$$

Then there exists a continuous and exponential bounded function κ in $[0, \infty)$ such that $\mathcal{L}\kappa = 1/K$. Moreover, $0 \in \operatorname{supp}(\kappa)$ and we apply Theorem 5.2.12 to conclude that $\mathcal{D}_\kappa \subsetneq \mathcal{D}_+$.

We note that for the cases $\kappa = \mathfrak{r}_\nu$ (corresponding to local integrated semigroups) and $\mathcal{L}\kappa(\lambda) = \displaystyle\prod_{j=1}^{\infty} \left(1 + \frac{lz}{j^{1/a}}\right)^{-1}$ (where $l > 0, 0 < a < 1$ and which are considered in [C] and will be presented in subsection 5.2.5 below), we have $\mathcal{L}\kappa(\lambda) \neq 0$ for all $\operatorname{Re} \lambda > \operatorname{abs}(|\kappa|)$.

5.2.3 Local convoluted semigroups

The definition of global κ-convoluted semigroups was introduced by the first time by I. Cioranescu [C] and subsequently developed in [CL] (see also [KMV] and the monographs [Ko] and [MF]). We will consider the following definition of local κ-convoluted semigroup as appears in [KP, Definition 2.1]

Definition 5.2.14. *Let $0 < \tau \leq \infty$, $\kappa \in L^1_{loc}([0, \tau))$ and A be a closed operator. Let furthermore $(S_\kappa(t))_{t \in [0,\tau)} \subset \mathcal{B}(X)$ be a strongly continuous operator family. The family $(S_\kappa(t))_{t \in [0,\tau)}$ is a local κ-convoluted semigroup (or local κ-semigroup in short) generated by A if $S_\kappa(t)A \subset AS_\kappa(t)$, $\int_0^t S_\kappa(s)x ds \in D(A)$ for $t \in [0, \tau)$ and $x \in X$ and*

$$(5.20) \qquad A \int_0^t S_\kappa(t) x dt = S_\kappa(t)x - \int_0^t \kappa(s) ds x, \quad x \in X,$$

for $t \in [0, \tau)$; in this case the operator A is called the generator of $(S_\kappa(t))_{t \in [0,\tau)}$. We say that $(S_\kappa(t))_{t \in [0,\tau)}$ is non degenerate if $S(t)x = 0$ for all $0 \leq t < \tau$ implies $x = 0$.

Alternatively, in relation to Problem (5.12), we note that when the problem is well posed in the sense that for every $x \in X$, there exists a unique solution $v \in C^1([0,\tau), X) \cap C([0,\tau), D(A))$, we set $S(t)x = v'(t)$, $0 \le t < \tau$, $x \in X$. It follows from the Closed Graph Theorem that $S(t) \in \mathcal{B}(X)$, $0 \le t < \tau$. Clearly, $t \mapsto S(t)$ is strongly continuous from $[0,\tau)$ to $\mathcal{B}(X)$. The local convoluted semigroup defined in this manner is necessarily non degenerate, due to the uniqueness assumption.

It is easy to prove that if A generates a κ-convoluted semigroup $(S_\kappa(t))_{t\in[0,\tau)}$, then $S_\kappa(0) = 0$ and $S_\kappa(t)x \in \overline{D(A)}$ for $t \in [0,\tau)$ and $x \in X$. See more details, for example in [KMV] and [KP].

Remarks 5.2.15. (1) For $\nu > 0$ and $\kappa = \mathfrak{r}_\nu$, we get ν-times integrated semigroups which were introduced in [Hi1]. The case $\nu \in \mathbb{N}$ appeared earlier. We follow the usual notation $(S_\nu(t))_{t\in[0,\tau)}$ for ν-times local integrated semigroups.
(2) If $C \in \mathcal{B}(X)$ is an injective operator and we set $\kappa(t) \equiv C$, $0 \le t < \tau$ then we recover the concept of local C-regularized semigroups were first sudied in [TO].
(3) One condition in the definition of local convoluted semigroup, equation (5.20) may be interpreted as a Duhamel formula for the abstract Cauchy problem. More precisely, if we are interested in the (non-homogeneous) initial value problem

$$\begin{cases} u'(t) = Au(t) + F(t),\ 0 \le t < \tau \\ u(0) = x \in X, \end{cases}$$

where F is an X-valued function and $\tau \in (0,\infty]$, and $K(\cdot)$ takes values in $\mathcal{B}(X)$ with the additional assumptions that $K(t)K(s) = K(s)K(t)$, $t, s \in [0,\tau)$; $AK(t)x = K(t)Ax$, $t \in [0,\tau)$, $x \in D(A)$, we can consider the regularized problem

$$\begin{cases} v'(t) = Av(t) + K(t)x + F_K(t),\ 0 \le t < \tau \\ v(0) = 0, \end{cases}$$

in which $F_K(t) = (K * F)(t) = \int_0^t K(t-s)F(s)ds$, $0 \le t < \tau$. More details can be found in the reference [CL]. We shall be concerned only with the situation where $K(t) = \phi(t)I$ where I is the identity operator on X. Spectral criteria for the generation of local convoluted semigroups involving the resolvent of the generator can be found in the references [C], [CL], [KMV] and [Ko].
(4) Other equivalent definitions of local convoluted semigroup, using the composition property (see Proposition 5.2.16) or the Laplace transform ([KMV, Theorem 3.2]) show this algebraic aspect in a straightforward way.

The next characterization of local κ-semigroups has the advantage to offer an algebraic character which is crucial in the development of the theory as we will see in Theorem 5.2.22. The proof runs parallel to the global case presented in [KP, Proposition 2.2], see also [Ko, Proposition 2.1.5].

Proposition 5.2.16. *Let $0<\tau\le\infty$, $\kappa\in L^1_{loc}([0,\tau))$, A a closed linear operator and $(S_\kappa(t))_{t\in[0,\tau)}$ be a non-degenerate strongly continuous operator family. Then $(S_\kappa(t))_{t\in[0,\tau)}$ is a local κ-convoluted semigroup generated by A if and only if $S_\kappa(0)=0$ and*

$$S_\kappa(t)S_\kappa(s)x=\int_t^{t+s}\kappa(t+s-r)S_\kappa(r)xdr-\int_0^s\kappa(t+s-r)S_\kappa(r)xdr,\qquad x\in X,\tag{5.21}$$

for $0\le s,t\le t+s<\tau$.

Note that if we take $\kappa\in L^1_{loc}([0,\tau))$, and we define $(\kappa_t)_{t\in[0,\tau)}\subset L^1_{loc}([0,\tau))$ by

$$\kappa_t(s):=\kappa(t-s)\chi_{[0,t]}(s),\qquad s\in[0,\tau),$$

then by Proposition 5.2.16 and Corollary 5.2.3, we may conclude that $(\kappa_t)_{t\in[0,\tau)}$ is a local κ-convoluted semigroup in $L^1_{loc}([0,\tau))$. In this section, we only consider local κ-convoluted semigroups which are non-degenerate.

The next theorem is the main result in this section and shows how a local κ-convoluted semigroup $(S_\kappa(t))_{t\in[0,\tau)}$ is extended to $[0,n\tau)$; in fact we get a local κ^{*n}-convoluted semigroup in $[0,n\tau)$ for $n\in\mathbb{N}$. Note that we improve previous results ([CL, Section 2] and [Ko, Theorem 2.1.1.9]): our approach is sharper that n-iterations of these theorems.

Theorem 5.2.17. *Let $n\in\mathbb{N}$, $0<\tau\le\infty$, $\kappa\in L^1_{loc}([0,(n+1)\tau))$ and $(S_\kappa(t))_{t\in[0,\tau)}$ be a local κ-convoluted semigroup generated by A. Then the family of operators $(S_{\kappa^{*(n+1)}}(t))_{t\in[0,(n+1)\rho]}$ defined by*

$$S_{\kappa^{*(n+1)}}(t)x=\int_0^t\kappa(t-s)S_{\kappa^{*n}}(s)xds,\qquad x\in X,$$

for $t\in[0,n\rho]$ and

$$S_{\kappa^{*(n+1)}}(t)x=S_{\kappa^{*n}}(n\rho)S_\kappa(t-n\rho)x+\int_0^{n\rho}\kappa(t-s)S_{\kappa^{*n}}(s)xds+\int_0^{t-n\rho}\kappa^{*n}(t-s)S_\kappa(s)xds,$$

for $x\in X$ and $t\in[n\rho,(n+1)\rho]$ is a local $\kappa^{*(n+1)}$-semigroup generated by A for any $\rho<\tau$. Then we conclude that A generates a local $\kappa^{*(n+1)}$-semigroup $(S_{\kappa^{*(n+1)}}(t))_{t\in[0,(n+1)\tau)}$.

Proof. Note that the family of operators $(S_{\kappa^{*(n+1)}}(t))_{t\in[0,(n+1)\rho]}$ is strongly continuous. It is known that $(S_{\kappa^{*(n+1)}}(t))_{t\in[0,n\rho]}$ is a local $\kappa^{*(n+1)}$-semigroup generated by A, see for example [Ko, Proposition 2.1.3] and [KLM, Proposition 5.2]. Now take $t\in[n\rho,(n+1)\rho]$ and $x\in X$. It is clear that $S_{\kappa^{*(n+1)}}(t)A\subset AS_{\kappa^{*(n+1)}}(t)$ and we show that $\int_0^t S_{\kappa^{*(n+1)}}(r)xdr\in D(A)$. Since

$$\int_0^t S_{\kappa^{*(n+1)}}(r)xdr=\int_0^{n\rho}S_{\kappa^{*(n+1)}}(r)xdr+\int_{n\rho}^t S_{\kappa^{*(n+1)}}(r)xdr,$$

we check that $\int_{n\rho}^t S_{\kappa^{*(n+1)}}(r)xdr \in D(A)$, i.e,

$$\int_{n\rho}^t \left(\int_0^{n\rho} \kappa(r-s) S_{\kappa^{*n}}(s)xds + \int_0^{r-n\rho} \kappa^{*n}(r-s) S_\kappa(s)xds \right) dr \\ + \int_{n\rho}^t S_{\kappa^{*n}}(n\rho) S_\kappa(r-n\rho)xdr \ \in \ D(A).$$

As $(S_\kappa(t))_{t\in[0,\tau)}$ is a local κ-convoluted semigroup generated by A, we get that

$$S_{\kappa^{*n}}(n\rho) \int_{n\rho}^t S_\kappa(r-n\rho)xdr \in D(A).$$

Now we prove that $\int_{n\rho}^t \int_0^{n\rho} \kappa(r-s) S_{\kappa^{*n}}(s)xdsdr \in D(A)$ as well. We apply Fubini's theorem and a change of variable $u = r - s$, to obtain that

$$\int_{n\rho}^t \int_0^{n\rho} \kappa(r-s) S_{\kappa^{*n}}(s)xdsdr = \int_0^{n\rho} S_{\kappa^{*n}}(s)x \int_{n\rho}^t \kappa(r-s)drds \\ = \int_0^{n\rho} S_{\kappa^{*n}}(s)x \int_{n\rho-s}^{t-s} \kappa(u)duds = \int_0^t \kappa(u) \int_{\max\{n\rho-u,0\}}^{\min\{t-u,n\rho\}} S_{\kappa^{*n}}(s)xdsdu \in D(A).$$

In a similar way, it is shown that

$$\int_{n\rho}^t \int_0^{r-n\rho} \kappa^{*n}(r-s) S_\kappa(s)xdsdr = \int_{n\rho}^t \kappa^{*n}(u) \int_0^{t-u} S_\kappa(s)xdsdu \in D(A).$$

To finish the proof we prove the equality (5.20) for $t \in [n\rho, (n+1)\rho]$ and $x \in X$. Note that

$$A\int_0^t S_{\kappa^{*(n+1)}}(s)xds = S_{\kappa^{*(n+1)}}(n\rho)x - \int_0^{n\rho} \kappa^{*(n+1)}(s)dsx + A\int_{n\rho}^t S_{\kappa^{*(n+1)}}(s)xds.$$

We apply Fubini's theorem to obtain that

$$\begin{aligned} \int_{n\rho}^t S_{\kappa^{*(n+1)}}(s)xds \quad &= \quad S_{\kappa^{*n}}(n\rho) \int_0^{t-n\rho} S_\kappa(u)xdu + \int_0^{n\rho} S_{\kappa^{*n}}(r)x \int_{n\rho}^t \kappa(s-r)dsdr \\ &\quad + \int_0^{t-n\rho} S_\kappa(r)x \int_{r+n\rho}^t \kappa^{*n}(s-r)dsdr. \end{aligned} \tag{5.22}$$

We apply the operator A to the first summand to get that

$$S_{\kappa^{*n}}(n\rho) A \int_0^{t-n\rho} S_\kappa(u)xdu = S_{\kappa^{*n}}(n\rho) S_\kappa(t-n\rho)x - S_{\kappa^{*n}}(n\rho)x \int_0^{t-n\rho} \kappa(u)du.$$

In the second summand of (5.22) we write

$$\int_{n\rho}^t \kappa(s-r)ds = \int_0^{t-r} \kappa(u)du - \int_0^{n\rho-r} \kappa(u)du.$$

Then we apply the operator A and the Fubini theorem to obtain that

$$\begin{aligned}
A\int_0^{n\rho} S_{\kappa^{*n}}(r)x\int_0^{t-r}\kappa(u)dudr &= A\int_0^t \kappa(u)\int_0^{\min\{n\rho,t-u\}} S_{\kappa^{*n}}(r)xdrdu\\
&= \left(S_{\kappa^{*n}}(n\rho)x-\int_0^{n\rho}\kappa^{*n}(y)dyx\right)\int_0^{t-n\rho}\kappa(u)du\\
&\quad+\int_{t-n\rho}^t \kappa(u)\left(S_{\kappa^{*n}}(t-u)x-\int_0^{t-u}\kappa^{*n}(y)dyx\right)du\\
&= S_{\kappa^{*n}}(n\rho)x\int_0^{t-n\rho}\kappa(u)du+\int_0^{n\rho}\kappa(t-r)S_{\kappa^{*n}}(r)xdr\\
&\quad-\left(\int_0^{n\rho}\kappa^{*n}(y)dy\right)\left(\int_0^{t-n\rho}\kappa(u)du\right)x-\int_0^{n\rho}\kappa(t-r)\int_0^r\kappa^{*n}(y)dydrx.
\end{aligned}$$

Using similar ideas we also get that

$$\begin{aligned}
A\int_0^{n\rho} S_{\kappa^{*n}}(r)x\int_0^{n\rho-r}\kappa(u)dudr &= \int_0^{n\rho}\kappa(u)\left(S_{\kappa^{*n}}(n\rho-u)x-\int_0^{n\rho-u}\kappa^{*n}(r)drx\right)du\\
&= S_{\kappa^{*(n+1)}}(n\rho)x-\int_0^{n\rho}\kappa^{*(n+1)}(r)drx.
\end{aligned}$$

In the third summand of (5.22) we write

$$\int_{r+n\rho}^t \kappa^{*n}(s-r)ds=\int_r^t-\int_r^{r+n\rho}\kappa^{*n}(s-r)ds.$$

We apply the operator A and the Fubini theorem to obtain that

$$\begin{aligned}
A\int_0^{t-n\rho} S_\kappa(r)x\int_r^t \kappa^{*n}(s-r)dsdr &= A\int_0^t \kappa^{*n}(u)\int_0^{\min\{t-n\rho,t-u\}} S_\kappa(r)xdrdu\\
&= S_\kappa(t-n\rho)x\int_0^{n\rho}\kappa^{*n}(u)du+\int_0^{t-n\rho}\kappa^{*n}(t-r)S_\kappa(r)xdr\\
&\quad-\left(\int_0^{n\rho}\kappa^{*n}(u)du\right)\left(\int_0^{t-n\rho}\kappa(u)du\right)-\int_0^{t-n\rho}\kappa^{*n}(t-r)\int_0^r\kappa(y)dy.
\end{aligned}$$

and finally we get

$$\begin{aligned}
&A\int_0^{t-n\rho} S_\kappa(r)x\int_r^{r+n\rho}\kappa^{*n}(s-r)dsdr\\
&\qquad= S_\kappa(t-n\rho)x\int_0^{n\rho}\kappa^{*n}(u)du-\left(\int_0^{n\rho}\kappa^{*n}(u)du\right)\left(\int_0^{t-n\rho}\kappa(u)du\right).
\end{aligned}$$

We conclude that

$$\begin{aligned}
&A\int_0^{t-n\rho} S_\kappa(r)x\int_{r+n\rho}^t \kappa^{*n}(s-r)dsdr\\
&\qquad= \int_0^{t-n\rho}\kappa^{*n}(t-r)S_\kappa(r)xdr-\int_0^{t-n\rho}\kappa^{*n}(t-r)\int_0^r\kappa(y)dy.
\end{aligned}$$

To complete the proof we put together all summands to have that

$$\begin{aligned}
A\int_0^t S_{\kappa^{*(n+1)}}(s)xds &= S_{\kappa^{*n}}(n\rho)S_\kappa(t-n\rho)x + \int_0^{n\rho}\kappa(t-r)S_{\kappa^{*n}}(r)xdr \\
&\quad + \int_0^{t-n\rho}\kappa^{*n}(t-r)S_\kappa(r)xdr - \left(\int_0^{n\rho}\kappa^{*n}(y)dy\right)\left(\int_0^{t-n\rho}\kappa(u)du\right)x \\
&\quad - \int_0^{n\rho}\kappa(t-r)\int_0^r\kappa^{*n}(y)dydrx - \int_0^{t-n\rho}\kappa^{*n}(t-r)\int_0^r\kappa(y)dydr \\
&= S_{\kappa^{*(n+1)}}(t)x - \int_0^t\kappa^{*(n+1)}(s)xds,
\end{aligned}$$

where we have used the Lemma 5.2.1. This proves the claim. □

In fact, the expression of the $(S_{\kappa^{*(n+1)}}(t))_{t\in[0,(n+1)\rho]}$ is not unique as shown in the next result. Both proofs are similar to the proof of Theorem 5.2.17 and are left to the reader.

Theorem 5.2.18. *Let $n\geq 2$, $0<\tau\leq\infty$, $\kappa\in L^1_{loc}([0,n\tau))$ and let $(S_\kappa(t))_{t\in[0,\tau)}$ be a local κ-convoluted semigroup with generator A. Then the family of operators $(S_{\kappa^{*n}}(t))_{t\in[0,n\rho]}$ defined in Theorem 5.2.17 satisfies*

$$S_{\kappa^{*n}}(t)x = \int_0^t\kappa^{*(n-j)}(t-s)S_{\kappa^{*j}}(s)xds, \qquad x\in X,$$

for $t\in[0,j\rho]$ and

$$\begin{aligned}
S_{\kappa^{*n}}(t)x &= S_{\kappa^{*j}}(n\rho)S_{\kappa^{*(n-j)}}(t-j\rho)x \\
&\quad + \int_0^{j\rho}\kappa^{*(n-j)}(t-s)S_{\kappa^{*j}}(s)xds + \int_0^{t-j\rho}\kappa^{*j}(t-s)S_{\kappa^{*(n-j)}}(s)xds,
\end{aligned}$$

for $x\in X$, $1\leq j\leq n-1$ and $t\in[j\rho,n\rho]$ and any $\rho<\tau$.

Corollary 5.2.19. *Let $0<\tau\leq\infty$, $\kappa\in L^1_{loc}([0,2\tau))$ and let $(S_\kappa(t))_{t\in[0,\tau)}$ be a local κ-convoluted semigroup. Then the above family of operators $(S_{\kappa*\kappa}(t))_{t\in[0,2\rho]}$ defined in Theorem 5.2.17 satisfies*

$$S_{\kappa*\kappa}(t+s)x = S_\kappa(t)S_\kappa(s)x + \left(\int_0^t + \int_0^s\right)\kappa(t+s-u)S_\kappa(u)xdu$$

for $t,s\in[0,\rho)$ and $x\in X$, for any $\rho<\tau$.

The next result was obtained in [Mi4, Theorem 2] in the case $n=1$.

Corollary 5.2.20. *Let $n\in\mathbb{N}$, $0<\tau\leq\infty$ and let $(S_\nu(t))_{t\in[0,\tau)}$ be a local ν-times integrated semigroup with generator A. Then the family of operators $(S_{(n+1)\nu}(t))_{t\in[0,(n+1)\rho]}$ defined by*

$$S_{(n+1)\nu}(t)x = \int_0^t\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}S_{n\nu}(s)xds, \qquad x\in X,$$

for $t \in [0, n\rho]$ *and*

$$S_{(n+1)\nu}(t)x = S_{n\nu}(n\rho)S_\nu(t-n\rho)x + \int_0^{n\rho} \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} S_{n\nu}(s)xds + \int_0^{t-n\rho} \frac{(t-s)^{n\nu-1}}{\Gamma(n\nu)} S_\nu(s)xds,$$

for $x \in X$ *and* $t \in [n\rho, (n+1)\rho]$ *is a local* ν*-times integrated semigroup generated by* A *for any* $\rho < \tau$.
Therefore we conclude that A *generates a local* ν*-times integrated semigroup on* $[0,(n+1)\tau)$, *namely* $(S_{(n+1)\nu}(t))_{t\in[0,(n+1)\tau)}$.

Remark 5.2.21. In the case $\nu \in \mathbb{N}$, and $t \in [n\rho, (n+1)\rho]$, note that

$$\int_0^{n\rho} \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} S_{n\nu}(s)xds = \sum_{j=0}^{\nu-1} \frac{(t-n\rho)^j}{j!} S_{(n+1)\nu-j}(n\rho)$$

and

$$\int_0^{t-n\rho} \frac{(t-s)^{n\nu-1}}{\Gamma(n\nu)} S_\nu(s)xds = \sum_{j=0}^{n\nu-1} \frac{(n\rho)^j}{j!} S_{(n+1)\nu-j}(t-n\rho)$$

for $x \in X$ and we recover the extension given in [AEK, Theorem 4.1] for the case $n = 1$.
To finish this subsection, we mention that other extension result (for scalar ν-times semigroups in this case) may be found in [Ku, Lemma 4.4],

$$I_t^n * I_s^n = I_{s+t}^{2n} - \sum_{j=0}^{n-1} \left(\frac{s^j}{j!} I_t^{2n-j} - \frac{t^j}{j!} I_s^{2n-j} \right), \qquad t, s > 0,$$

where $I_t^n(r) := \frac{(t-r)^n}{n!}\chi_{[0,t]}(r)$, $r \in \mathbb{R}^+$ and $n \in \mathbb{N} \cup \{0\}$.

5.2.4 Algebra homomorphisms and local convoluted semigroups

As the following theorem shows, local κ-convoluted semigroups induce algebra homomorphism from certain spaces of test functions $\mathcal{D}_{\kappa^{*\infty}}$. Note that the extension theorem (Theorem 5.2.17) is necessary to define the algebra homomorphisms from functions defined on $\mathbb{R}^+$. The space $\mathcal{D}_{\kappa^{*\infty}}$ is introduced in Definition 5.2.9.

Theorem 5.2.22. *Let* $\kappa \in L^1_{loc}(\mathbb{R}^+)$ *with* $0 \in supp(\kappa)$, *and let* $(S_\kappa(t))_{t\in[0,\tau]}$ *be a non-degenerate local* κ*-convoluted semigroup generated by* A. *We define the map* $\mathcal{G}_\kappa : \mathcal{D}_{\kappa^{*\infty}} \to \mathcal{B}(X)$ *by*

$$\mathcal{G}_\kappa(f)x := \int_0^{n\tau} W_{\kappa^{*n}} f(t) S_{\kappa^{*n}}(t)xdt, \qquad x \in X, f \in \mathcal{D}_{\kappa^{*\infty}},$$

where $supp(f) \subset [0, n\tau]$ *and* $(S_{\kappa^{*n}}(t))_{t\in[0,n\tau]}$ *is defined in Theorem 5.2.17 for some* $n \in \mathbb{N}$. *Then the following properties hold.*

(1) *The map* $\mathcal{G}_\kappa$ *is well defined, linear and bounded.*

(2) $\mathcal{G}_\kappa(f * g) = \mathcal{G}_\kappa(f)\mathcal{G}_\kappa(g)$ *for* $f, g \in \mathcal{D}_{\kappa^{*\infty}}$.

(3) $\mathcal{G}_\kappa(f)x \in D(A)$ *and* $A\mathcal{G}_\kappa(f)x = -\mathcal{G}_\kappa(f')x - f(0)x$ *for any* $f \in \mathcal{D}_{\kappa^{*\infty}}$ *and* $x \in X$.

Proof. Take $f \in \mathcal{D}_{\kappa^{*\infty}}$ and $\operatorname{supp}(f) \subset [0, n\tau]$ for some $n \in \mathbb{N}$. First, we prove that $\mathcal{G}_\kappa$ is well defined. Let $m \geq n$, $\kappa^{*m} = \kappa^{*n} * \kappa^{*(m-n)}$, and observe that $\kappa^{*(m-n)} \circ W_{\kappa^{*m}} f = W_{\kappa^{*n}} f$ by Lemma 5.2.8 (2). Now we apply the Lemma 5.2.7 to conclude $\operatorname{supp}(W_{\kappa^{*m}} f) \subset [0, n\tau]$. By Theorem 5.2.18 and the Fubini theorem, we get that

$$\begin{aligned}\int_0^{m\tau} W_{\kappa^{*m}} f(t) S_{\kappa^{*m}}(t)x dt &= \int_0^{n\tau} W_{\kappa^{*m}} f(t)(\kappa^{*(m-n)} * S_{\kappa^{*n}})(t)x dt \\ &= \int_0^{n\tau} \kappa^{*(m-n)} \circ W_{\kappa^{*m}} f(t) S_{\kappa^{*n}}(t)x dt = \int_0^{n\tau} W_{\kappa^{*n}} f(t) S_{\kappa^{*n}}(t)x dt\end{aligned}$$

for $x \in X$.

It is straightforward to check that the map $\mathcal{G}_\kappa$ is linear. Now take $(f_n)_{n\geq 1} \subset \mathcal{D}_{\kappa^{*\infty}}$, and $f \in \mathcal{D}_{\kappa^{*\infty}}$ such that $f_n \to f$. Then there exists $n \in \mathbb{N}$ such that $\operatorname{supp}(f_n), \operatorname{supp}(f) \subset [0, n\tau]$. Note that the map $t \mapsto S_{\kappa^{*n}}(t)x$, $[0, n\tau] \to X$, is continuous and

$$\begin{aligned}\|\mathcal{G}_\kappa(f_n)x - \mathcal{G}_\kappa(f)x\| &\leq \int_0^{n\tau} |W_{\kappa^{*n}} f_n(t) - W_{\kappa^{*n}} f(t)| \|S_{\kappa^{*n}}(t)x\| dt \\ &\leq C_x \int_0^{n\tau} |W_{\kappa^{*n}}(f_n - f)(t)| dt\end{aligned}$$

for $x \in X$. Now consider the operator $T'_{\kappa^{*n}} : L_1[0, n\tau] \to L_1[0, n\tau]$, $f \mapsto T'_{\kappa^{*n}}(f) = \kappa^{*n} \circ f$ given in subsection 5.2.2. By the open mapping theorem, $T'_{\kappa^{*n}}$ is open; we conclude that $W_{\kappa^{*n}} f_n \to W_{\kappa^{*n}} f$ in $L_1[0, n\tau]$, the map $\mathcal{G}_\kappa$ is bounded and the part (1) is proved.

Take $f, g \in \mathcal{D}_{\kappa^{*\infty}}$, i.e. $f, g \in \mathcal{D}_{\kappa^{*n}}$ and then $f * g \in \mathcal{D}_{\kappa^{*n}}$ (see [KLM, Theorem 2.10]) for $n \geq 1$. Then $f * g \in \mathcal{D}_{\kappa^{*\infty}}$. Now we show that $\mathcal{G}_\kappa(f * g) = \mathcal{G}_\kappa(f)\mathcal{G}_\kappa(g)$. Take $n \in \mathbb{N}$ such that $\operatorname{supp}(f), \operatorname{supp}(g) \subset [0, n\tau]$ and by Lemma 5.2.7, $\operatorname{supp}(W_{\kappa^{*2n}} f), \operatorname{supp}(W_{\kappa^{*2n}} g) \subset [0, n\tau]$. Then $\operatorname{supp}(f * g) \subset [0, 2n\tau]$ and $\operatorname{supp}(W_{\kappa^{*2n}}(f * g)) \subset [0, 2n\tau]$. By (5.18) we have that

$$\begin{aligned}\mathcal{G}_\kappa(f * g)x &= \int_0^{2n\tau} W_{\kappa^{*2n}}(f * g)(t) S_{\kappa^{*2n}}(t)x dt \\ &= \int_0^{2n\tau} \int_0^t W_{\kappa^{*2n}} g(r) \int_{t-r}^t \kappa^{*2n}(s + r - t) W_{\kappa^{*2n}} f(s) ds dr S_{\kappa^{*2n}}(t)x dt \\ &\quad - \int_0^{2n\tau} \int_t^{2n\tau} W_{\kappa^{*2n}} g(r) \int_t^{2n\tau} \kappa^{*2n}(s + r - t) W_{\kappa^{*2n}} f(s) ds dr S_{\kappa^{*2n}}(t)x dt.\end{aligned}$$

By repeated application of Fubini's theorem, we have:

$$\begin{aligned}\mathcal{G}_\kappa(f*g)x = {} & \int_0^{2n\tau} W_{\kappa^{*2n}}g(r)\int_0^r W_{\kappa^{*2n}}f(s)\int_r^{s+r}\kappa^{*2n}(s+r-t)S_{\kappa^{*2n}}(t)xdtdsdr \\ & +\int_0^{2n\tau} W_{\kappa^{*2n}}g(r)\int_r^{2n\tau} W_{\kappa^{*2n}}f(s)\int_s^{s+r}\kappa^{*2n}(s+r-t)S_{\kappa^{*2n}}(t)xdtdsdr \\ & -\int_0^{2n\tau} W_{\kappa^{*2n}}g(r)\int_0^r W_{\kappa^{*2n}}f(s)\int_0^{s}\kappa^{*2n}(s+r-t)S_{\kappa^{*2n}}(t)xdtdsdr \\ & -\int_0^{2n\tau} W_{\kappa^{*2n}}g(r)\int_r^{2n\tau} W_{\kappa^{*2n}}f(s)\int_0^{r}\kappa^{*2n}(s+r-t)S_{\kappa^{*2n}}(t)xdtdsdr \\ = {} & \int_0^{2n\tau} W_{\kappa^{*2n}}g(r)\int_0^r W_{\kappa^{*2n}}f(s)\left(\int_r^{s+r}-\int_0^s\kappa^{*2n}(s+r-t)S_{\kappa^{*2n}}(t)xdt\right)dsdr \\ & +\int_0^{2n\tau} W_{\kappa^{*2n}}g(r)\int_r^{n\tau} W_{\kappa^{*2n}}f(s)\left(\int_s^{s+r}-\int_0^r\kappa^{*2n}(s+r-t)S_{\kappa^{*2n}}(t)xdt\right)dsdr \\ = {} & \int_0^{2n\tau} W_{\kappa^{*2n}}g(r)S_{\kappa^{*2n}}(r)\int_0^r W_{\kappa^{*2n}}f(s)S_{\kappa^{*2n}}(s)xdsdr \\ & +\int_0^{2n\tau} W_{\kappa^{*2n}}g(r)S_{\kappa^{*2n}}(r)\int_r^{n\tau} W_{\kappa^{*2n}}f(s)S_{\kappa^{*2n}}(s)xdsdr \\ = {} & \int_0^{2n\tau} W_{\kappa^{*2n}}g(r)S_{\kappa^{*2n}}(r)\int_0^{2n\tau} W_{\kappa^{*2n}}f(s)S_{\kappa^{*2n}}(s)xdsdr = \mathcal{G}_\kappa(g)\mathcal{G}_\kappa(f)x,\end{aligned}$$

where we have applied formula (5.21). Then part (2) is proved.

Now we consider $f \in \mathcal{D}_{\kappa^{*\infty}}$, $\mathrm{supp}(f) \subset [0, n\tau]$ and $x \in X$. We apply the formulae (5.16) and (5.20) to get

$$\begin{aligned}A\mathcal{G}_\kappa(f)x &= -A\int_0^{n\tau}(W_{\kappa^{*n}}f)'(t)\int_0^t S_{\kappa^{*n}}(s)xdsdt \\ &= -\int_0^{n\tau}W_{\kappa^{*n}}f'(t)\left(S_{\kappa^{*n}}(t)x-\int_0^t\kappa^{*n}(s)dsx\right)dt \\ &= -\mathcal{G}_{\kappa^{*n}}(f')x-\int_0^{n\tau}W_{\kappa^{*n}}f(t)\kappa^{*n}(t)dtx \;=\; -\mathcal{G}_{\kappa^{*n}}(f')x-f(0)x,\end{aligned}$$

and the part (3) is proved. □

The previous theorem allows to show as a consequence one of main results in [KLM].

Remark 5.2.23. When the operator A generates a global convoluted semigroup $(S_\kappa(t))_{t\geq 0}$, the homomorphism $\mathcal{G}_\kappa$ is defined from $\mathcal{D}_\kappa$ to $\mathcal{B}(X)$ by

$$\mathcal{G}_\kappa(f)x = \int_0^\infty W_\kappa f(t)S_\kappa(t)xdt, \qquad x \in X, \qquad f \in \mathcal{D}_\kappa,$$

see [KLM, Theorem 5.5]. Under some conditions of the growth of $(S_\kappa(t))_{t\geq 0}$ (e.g. exponential, polynomial boundedness), the homomorphism $\mathcal{G}_\kappa$ may be extended to a bounded Banach algebra homomorphism, see [KLM, Theorem 5.6].

Let $\kappa, l \in L^1_{loc}([0,\tau))$ with $0 \in \operatorname{supp}(\kappa)$, and $(S_\kappa(t))_{t\in[0,\tau)}$ a non-degenerate local κ-convoluted semigroup generated by A. Then $(l * S_\kappa(t))_{t\in[0,\tau]}$ is a non-degenerate local κ-convoluted semigroup generated by A, see a similar proof in [KLM, Proposition 5.2].

Corollary 5.2.24. *Let $\kappa, l \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(\kappa) \cap supp(l)$, and let $(S_\kappa(t))_{t\in[0,\tau)}$, resp. $(S_{\kappa * l}(t))_{t\in[0,\tau)}$ non-degenerate local κ-convoluted, resp. $l * \kappa$-convoluted semigroups generated by A. Then*

$$\mathcal{G}_{\kappa * l}(f) = \mathcal{G}_\kappa(f), \qquad f \in \mathcal{D}_{(\kappa * l)^{*\infty}},$$

where $\mathcal{G}_\kappa, \mathcal{G}_{\kappa * l}$ are defined in Theorem 5.2.22.

Proof. Take $f \in \mathcal{D}_{(\kappa * l)^{*\infty}}$ and $\operatorname{supp}(f) \subset [0, n\tau]$. It follows from (5.17) that $f \in \mathcal{D}_{(\kappa * l)^{*\infty}} \subset \mathcal{D}_{\kappa^{*\infty}}$ and

$$\begin{aligned}
\mathcal{G}_{\kappa * l}(f)x &= \int_0^{n\tau} W_{(\kappa * l)^{*n}} f(t) S_{(\kappa * l)^{*n}}(t)x\,dt = \int_0^{n\tau} W_{\kappa^{*n} * l^{*n}} f(t) S_{\kappa^{*n} * l^{*n}}(t)x\,dt \\
&= \int_0^{n\tau} W_{\kappa^{*n} * l^{*n}} f(t) (l^{*n} * S_{\kappa^{*n}})(t)x\,dt = \int_0^{n\tau} (l^{*n} \circ W_{\kappa^{*n} * l^{*n}} f)(t) S_{\kappa^{*n}}(t)x\,dt \\
&= \int_0^{n\tau} W_{\kappa^{*n}} f(t) S_{\kappa^{*n}}(t)x\,dt = \mathcal{G}_\kappa(f)x
\end{aligned}$$

where we have applied formula (5.17). □

Corollary 5.2.25. *Let $(S_\nu(t))_{t\in[0,\tau]}$ be a non-degenerate local ν-times integrated semigroup generated by A. We define the map $\mathcal{G}_\nu : \mathcal{D}_+ \to \mathcal{B}(X)$ by*

$$\mathcal{G}_\nu(f)x := \int_0^{n\tau} W^{\nu n} f(t) S_{\nu n}(t)x\,dt, \qquad x \in X, f \in \mathcal{D}_+,$$

where $supp(f) \subset [0, n\tau]$ and $(S_{\nu n}(t))_{t\in[0,n\tau]}$ is defined in Theorem 5.2.17 for some $n \in \mathbb{N}$. Then the map $\mathcal{G}_\nu$ is well defined, linear, bounded and $\mathcal{G}_\nu(f * g) = \mathcal{G}_\nu(f)\mathcal{G}_\nu(g)$ for $f, g \in \mathcal{D}_+$. Moreover, $\mathcal{G}_\nu(f)x \in D(A)$ and

$$A\mathcal{G}_\nu(f)x = -\mathcal{G}_\nu(f')x - f(0)x, \qquad f \in \mathcal{D}_+, \quad x \in X.$$

5.2.5 Examples and applications

In this subsection we consider different examples of convoluted semigroups which have been presented in the literature. Our results are applied in all these examples to illustrate their importance and usefulness.

Differential operators on $L_p(\mathbb{R}^n)$

Let E be one of the spaces $L_p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, $C_0(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$, or $C_b(\mathbb{R}^n)$ and A_E the associated operator to a partial differential operator with constant coefficients and defined by Fourier multipliers, see details in [Hi1, Section 4], [ABHN, Chapter 8]. Under some conditions, the operator A_E generates an ν-times integrated semigroup

$(S_\nu(t))_{t\geq 0}$ on E and $\|S_\nu(t)\| \leq C_E(1+t^\nu)$ for some constant C_E and certain $\nu > 0$, see [AK, Theorem 6.3] and [Hi1, Theorem 4.2]. In this case the map $\mathcal{G}_\nu : \mathcal{D}_+ \to \mathcal{B}(L_p(\mathbb{R}^N))$ (Corollary 5.2.25) extends to a Banach algebra homomorphism $\mathcal{G}_\nu : \mathcal{T}^\nu(1+t^\nu) \to \mathcal{B}(L_p(\mathbb{R}^N))$ where $\mathcal{T}^\nu(1+t^\nu)$ is the completion of $\mathcal{D}_+$ in the norm

$$\|f\| := \int_0^\infty |W^\nu f(t)|(1+t^\nu)dt, \qquad f \in \mathcal{D}_+,$$

where W^ν is the Weyl derivation of order ν, see [Mi1, Proposition 4.7]. Other examples of global ν-times integrated semigroups may be found in [CCO, Theorem 5.3] and [KW, Proposition 8.1].

Multiplication local integrated semigroup in ℓ^2

Let ℓ^2 be the Hilbert space of all complex sequences $x = (x_m)_{m=1}^\infty$ such that

$$\sum_{m=1}^\infty |x_m|^2 < \infty,$$

with the euclidean norm $\|x\| := \left(\sum_{m=1}^\infty |x_m|^2\right)^{\frac{1}{2}}$. Take $T > 0$ and define

$$a_m = \frac{m}{T} + i\left(\left(\frac{e^m}{m}\right)^2 - \left(\frac{m}{T}\right)^2\right)^{\frac{1}{2}}, \quad m \in \mathbb{N},$$

where $i^2 = -1$. For any $\nu > 0$ let $(S_\nu(t))_{t>0}$ be defined by

$$S_\nu(t)x = \left(\frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} e^{a_m s} x_m ds\right)_{m=1},$$

for $x \in D(S_\nu(t))$ where $D(S_\nu(t)) = \{x \in \ell^2 \,; S_\nu(t)x \in \ell^2\}$. Then the family $(S_\nu(t))_{t\in[0,\nu T)}$ is a local ν-times integrated semigroup on ℓ^2, $(S_\nu(t))_{t\in[0,\nu T)} \subset \mathcal{B}(\ell^2)$, such that $(S_\nu(t))_{t\in[0,\nu T)}$ cannot be extended to $t \geq \nu T$, see [Mi4, Example 1], [MF, Example 1.2.6]. We may apply Corollary 5.2.20 to define $(S_{n\nu}(t))$ for $t < n\nu T$ and Corollary 5.2.25 to define the map $\mathcal{G}_\nu : \mathcal{D}_+ \to \mathcal{B}(X)$ by

$$\mathcal{G}_\nu(f)x := \int_0^{n\tau} W^{\nu n} f(t) S_{\nu n}(t)x dt, \qquad x \in X, f \in \mathcal{D}_+,$$

with $\mathrm{supp}(f) \subset [0, n\tau]$ (with $\tau < T$). Other examples of local integrated semigroups defined by multiplication may be found in the reference [AEK, Example 4.4 (c), (d)].

The Laplacian on $L_2[0,\pi]$ with Dirichlet boundary conditions

The operator $-\Delta$ on $L_2[0,\pi]$ with Dirichlet or Neumann boundary conditions generates a polynomially bounded κ-convoluted semigroups $(S_\kappa(t))_{t\geq 0}$, where $\|S_\kappa(t)\| \leq C(1+t^3)$

where $C > 0$; note that κ is given in Example 5.2.13 and $|\kappa(t)| \leq Ce^{\beta}t$ for $t \geq 0$ and some $C, \beta > 0$, [B, Section 3, 5], [KP, Example 6.1]. By Remarks 5.2.23, there exists an algebra homomorphism $\mathcal{G}_\kappa : \mathcal{D}_\kappa \to \mathcal{B}(L_2[0,\pi])$ such that it extends to a Banach algebra homomorphism $\mathcal{G}_\kappa : \mathcal{T}_\kappa(e_\beta) \to \mathcal{B}(L_2[0,\pi])$, see Theorem 5.2.11 and [KLM, Theorem 6.5]. This example corresponds to the backward heat equation.

Other examples of generators of κ-convoluted semigroups (which does not generate integrated semigroups) may be found in [KP, Example 6.2], or [Ko, Section 2.8], where

$$\kappa(t) = \frac{e^{-a^2/4t}}{2\sqrt{\pi t^3}} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\lambda t/a^2 - \sqrt{\lambda}} d\lambda, \qquad t > 0,$$

for some $a > 0$ (where in the integral, r is any positive real number). In this case, we have:

$$\mathcal{L}\kappa(\lambda) = \frac{1}{a}e^{-a\sqrt{\lambda}}, \quad \operatorname{Re}\lambda > 0.$$

Ultradistributions in the Gevrey classes

Let M_j, $j = 0, 1, 2, \ldots$ be a Gevrey type sequences, i.e., a sequence of positive numbers such that $M_0 = 1$, which is logarithmically convex and non-quasianalytic, namely:

$$M_j^2 \leq M_{j-1}M_{j+1}, \quad \text{and} \quad \sum_{j=1}^{\infty} M_{j-1}/M_j < \infty,$$

for example $(j!^s)$, (j^{js}) and $\Gamma(1 + js)$ for $s > 1$. Let $m_j = M_j/M_{j-1}$ for $j \in \mathbb{N}$,

$$P(z) = \prod_{j=1}^{\infty}\left(1 + \frac{z}{m_j}\right), \qquad \operatorname{Re} z > 0,$$

and the function K defined by $\mathcal{L}K = P$, [CL, Section 5. Applications], [C, 4. Example and final comments].

The entire function $P(z)$ is called an ultradifferential polynomial. The following two operators are generators of local K-convoluted semigroups and appear in [C, 4. Example and final comments]:

(1) Let $X = L_2(\mathbb{R})$ and $A = i\frac{d^4}{dt^4} - \frac{d^2}{dt^2}$. Then A generates an K_1-convoluted semigroup on $[0, \tau)$, where $\mathcal{L}K_1 = P_1$ where $P_1(z) = \prod_{j=1}^{\infty}\left(1 + \frac{lz}{j^2}\right)$ for some $l > 0$.

In the context of ultradistribution semigroups, this example was first proposed by J. Chazarain ([Ch, Remarque 6.4]). It was observed in [K] that one can take $A = iB$ where B is the generator of a strongly continuous cosine function. Additional results are given in [Ko], [KP] including the case of Beurling ultradistributions. Other developments are discussed in [Ko], including a generalization of the abstract Weierstrass formula.

(2) Let μ be a σ-finite measure, the Lebesgue space $X = L_p(\Omega)$ $(1 \leq p \leq \infty)$ and $m : \Omega \to \mathbb{C}$ a measurable function. We define the multiplication operator A by $Af = mf$ where $D(A) = \{f \in L_p(\Omega); mf \in L_p(\Omega)\}$ and

$$\{z \in \mathbb{C}; \operatorname{Re} z \geq \alpha|z|^a + \beta\} \subset \rho(A).$$

for $\alpha, \beta > 0$ and $0 < a < 1$. For every $\tau > 0$ there is $l > 0$ such that A generates a local K_2-convoluted semigroup on $[0, \tau)$ where

$$P_2(z) = \prod_{j=1}^{\infty} \left(1 + \frac{lz}{j^{\frac{1}{a}}}\right),$$

where $\mathcal{L}K_2 = P_2$.

The following estimates are valid for the Gevrey sequences $M_j = (j!^s)$, $M_j = (j^{js})$ and $M_j = \Gamma(1 + js)$ where $s > 1$ is given.

$$e^{(l|z|)^a} \leq |P(z)| \leq e^{(L|z|)^a}, \quad \operatorname{Re} z \geq 0,$$

where L is a positive constant, see e.g. [C, (1.1)].

We apply Theorem 5.2.17 to conclude that A generates a local K^{*n}-convoluted semigroup on $[0, n\tau)$.

5.2.6 κ-Distribution Semigroups

Let X be a Banach space. Vector valued algebraic distributions, i.e., linear and continuous maps from a test function space to the space of bounded linear operators, $\mathcal{B}(X)$, which satisfy an algebraic property (similar to Theorem 5.2.22 (ii)) have been studied deeply in a large number of papers, see [F, Chapter 8]. In this sense, distribution semigroups (*in the sense of Lions*, DS-L) were introduced by J.L. Lions in [Li], see also [Ch], [AEK, Definition 7.1]. P.C. Kunstmann considered pre-distribution semigroups (or quasidistribution semigroup in the terminology of S. W. Wang) as linear and continuous maps $\mathcal{G} : \mathcal{D} \mapsto \mathcal{B}(X)$, $\mathcal{G} \in \mathcal{D}'(\mathcal{B}(X))$, satisfying

(1) $\mathcal{G}(\phi * \psi) = \mathcal{G}(\phi)\mathcal{G}(\psi)$ for $\phi, \psi \in \mathcal{D}$,

(2) $\cap\{\ker(\mathcal{G}(\theta)) \mid \theta \in \mathcal{D}_0\} = \{0\}$,

see [Ku, Definition 2.1] and [W, Definition 3.3]. In fact, a pre-distribution $\mathcal{G}$ can be regarded as a continuous linear map from $\mathcal{D}_+$ into $\mathcal{B}(X)$, $\mathcal{G} : \mathcal{D}_+ \mapsto \mathcal{B}(X)$, such that

(1) $\mathcal{G}(\phi * \psi) = \mathcal{G}(\phi)\mathcal{G}(\psi)$ for $\phi, \psi \in \mathcal{D}_+$.

(2) $\cap\{\ker(\mathcal{G}(\theta)) \mid \theta \in \mathcal{D}_+\} = \{0\}$.

see [W, Remark 3.4]. The differences between quasi-distribution and distribution semigroup in the sense of Lions may be found in [Ku, Remark 3.13].

Classes of Distribution semigroups on $(0,\infty)$ (in short DS on $(0,\infty)$) were considered in [KuMiP, Definition 1]. The subspace $\mathcal{D}'_+(\mathcal{B}(X)) \subset \mathcal{D}'(\mathcal{B}(X))$ is formed of the elements supported in $[0,\infty)$. A distribution semigroup on $(0,\infty)$, $\mathcal{G} \in \mathcal{D}'_+(\mathcal{B}(X))$, is a continuous linear map from $\mathcal{D}$ into $\mathcal{B}(X)$ say, $\mathcal{G} : \mathcal{D} \mapsto \mathcal{B}(X)$, supported in $[0,\infty)$, such that

(1) $\mathcal{G}(\phi * \psi) = \mathcal{G}(\phi)\mathcal{G}(\psi)$ for $\phi, \psi \in \mathcal{D}_0$.

(2) $\cap\{\ker(\mathcal{G}(\theta)) \mid \theta \in \mathcal{D}_0\} = \{0\}$.

Distribution semigroups on $(0,\infty)$ and quasi-distribution semigroups are not the same concept, see [KuMiP, Remark 4]. However, a particular class of distribution semigroups on $(0,\infty)$ (called distribution semigroups, see [KuMiP, Definition 2]) may be identified with quasi-distribution semigroups, [KuMiP, Theorem 1].

Keeping in mind these definitions and Theorem 5.2.22, we introduce the concept of κ-distribution semigroup.

Definition 5.2.26. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in supp(\kappa)$. We say that a linear and continuous map $\mathcal{G}_\kappa : \mathcal{D}_{\kappa^*\infty} \mapsto \mathcal{B}(X)$ is a κ-distribution semigroup, in short κ-DS, if it satisfies the following two conditions.*

(1) $\mathcal{G}_\kappa(\phi * \psi) = \mathcal{G}_\kappa(\phi)\mathcal{G}_\kappa(\psi)$ *for* $\phi, \psi \in \mathcal{D}_{\kappa^*\infty}$.

(2) $\cap\{\ker(\mathcal{G}_\kappa(\theta)) \mid \theta \in \mathcal{D}_{\kappa^*\infty}\} = \{0\}$.

In the case that $\mathcal{G}_\kappa$ is a κ-distribution semigroup on $(0,\infty)$, then $\mathcal{D}_{\kappa^*\infty} \neq \{0\}$ by (2).

Remark 5.2.27. Let $\mathcal{G} : \mathcal{D} \to \mathcal{B}(X)$ be a distribution semigroup (or quasi-distribution semigroup in the sense of Wang). Then $\mathcal{G} \circ \Lambda$ is a κ-distribution semigroup for any $\kappa \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in \mathrm{supp}(\kappa)$ and $\mathcal{D}_{\kappa^*\infty} = \mathcal{D}_+$; in particular $\mathcal{G} \circ \Lambda$ is a $\mathfrak{r}_\nu - DS$ for any $\nu > 0$ (see definition of Λ in subsection 5.2.2).

For a given κ-DS $\mathcal{G}_\kappa$, define the operator A' by

(i) $D(A') := \cup\{\mathrm{Im}(\mathcal{G}_\kappa(\theta)) \mid \theta \in \mathcal{D}_{\kappa^*\infty}\}$.

(ii) $A'\mathcal{G}_\kappa(\theta)x := -\mathcal{G}_\kappa(\theta')x - \theta(0)x$, for $x \in X$ and $\theta \in \mathcal{D}_{\kappa^*\infty}$.

Proposition 5.2.28. *The operator A' is well defined and closable.*

Proof. Assume that $\mathcal{G}_\kappa(\phi)x = \mathcal{G}_\kappa(\psi)y$ for some $x, y \in X$ and $\phi, \psi \in \mathcal{D}_{\kappa^*\infty}$. Now take $\theta \in \mathcal{D}_{\kappa^*\infty}$. Since

$$\theta * \phi'(t) = \theta' * \phi(t) + \theta(0)\phi(t) - \phi'(0)\theta(t), \qquad t \geq 0,$$

see, for example [W, Proposition 3.1 (iii)], we get that

$$\mathcal{G}_\kappa(\theta)\mathcal{G}_\kappa(\phi')x = \mathcal{G}_\kappa(\theta')\mathcal{G}_\kappa(\phi)x + \theta(0)\mathcal{G}_\kappa(\phi)x - \phi'(0)\mathcal{G}_\kappa(\theta)x$$

and hence

$$\left(-\mathcal{G}_\kappa(\theta') - \theta(0)\right)\mathcal{G}_\kappa(\phi)x = \mathcal{G}_\kappa(\theta)\left(-\mathcal{G}_\kappa(\phi')x - \phi'(0)x\right).$$

Similarly,

$$\left(-\mathcal{G}_\kappa(\theta') - \theta(0)\right)\mathcal{G}_\kappa(\psi)y = \mathcal{G}_\kappa(\theta)\left(-\mathcal{G}_\kappa(\psi')y - \psi'(0)y\right).$$

By Definition 5.2.26 (2), we conclude that

$$-\mathcal{G}_\kappa(\phi')x - \phi'(0)x = -\mathcal{G}_\kappa(\psi')y - \psi'(0)y$$

and A' is well defined.

To prove that A' is closable, let $(x_n)_{n\geq 1} \subset D(A')$ be such that $x_n \to 0$, $A'x_n \to y$. We write $x_n = \mathcal{G}_\kappa(\phi_n)z_n$ with $(\phi_n)_{n\geq 1} \subset \mathcal{D}_{\kappa^*\infty}$ and $(z_n)_{n\geq 1} \subset X$. Take $\theta \in \mathcal{D}_{\kappa^*\infty}$ and then

$$\begin{aligned}
\mathcal{G}_\kappa(\theta)y &= \lim_{n\to\infty} \mathcal{G}_\kappa(\theta)A'x_n = \lim_{n\to\infty} \mathcal{G}_\kappa(\theta)A'\mathcal{G}_\kappa(\phi_n)z_n \\
&= \lim_{n\to\infty} \mathcal{G}_\kappa(\theta)\left(-\mathcal{G}_\kappa(\phi_n')z_n - \phi_n(0)z_n\right) \\
&= \lim_{n\to\infty} \left(-\mathcal{G}_\kappa(\theta') - \theta(0)\right)\mathcal{G}_\kappa(\phi_n)z_n = \lim_{n\to\infty}\left(-\mathcal{G}_\kappa(\theta') - \theta(0)\right)x_n = 0.
\end{aligned}$$

This implies that $y = 0$ by Definition 5.2.26 (2). □

Definition 5.2.29. *The closure of A', denoted by A, is called the generator of $\mathcal{G}_\kappa$.*

Other definitions of generators of distribution semigroups are given using approximate units (see [AEK, Definition 7.1]) or the distribution $-\delta_0'$ ([Ku, Definition 3.3] and [KuMiP, Proposition 1]). In our case, given a κ-DS $\mathcal{G}_\kappa$ and its generator $(A, D(A))$, then

$$D(A) \subset \{x \in X |\ \text{exists } y \in X \text{ such that } \mathcal{G}_\kappa(\theta)y = -\mathcal{G}_\kappa(\theta')x - \theta(0)x \text{ for any } \theta \in \mathcal{D}_{\kappa^*\infty}\};$$

$$Ax = y, \qquad x \in D(A).$$

Theorem 5.2.30. *Let $\kappa \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(\kappa)$, $(S_\kappa(t))_{t\in[0,\tau]}$ a non-degenerate local κ-convoluted semigroup generated by A and $\mathcal{G}_\kappa : \mathcal{D}_{\kappa^*\infty} \to \mathcal{B}(X)$ the map defined in Theorem 5.2.22. Then $\mathcal{G}_\kappa$ is a κ-DS generated by A.*

Proof. The first condition in Definition 5.2.26 appears in Theorem 5.2.22 (2). Take $x \in X$ such that $\mathcal{G}_\kappa(\phi)(x) = 0$ for any $\phi \in \mathcal{D}_{\kappa^*\infty}$. Since $\phi' \in \mathcal{D}_{\kappa^*\infty}$, we apply Theorem 5.2.22 (3) to conclude $0 = \phi(0)x$ for any $\phi \in \mathcal{D}_{\kappa^*\infty}$ and then $0 = \phi_u(0)x$ for any $u \geq 0$. We conclude that $0 = \phi(u)x$ for any $u \geq 0$ and $\phi \in \mathcal{D}_{\kappa^*\infty}$. Then $x = 0$ and the condition Definition 5.2.26 (2) holds.

Note that $A\mathcal{G}_\kappa(\theta)x := -\mathcal{G}_\kappa(\theta')x - \theta(0)x$, for $x \in X$ and $\theta \in \mathcal{D}_{\kappa^*\infty}$, see Theorem 5.2.22 (3). As A is a closed operator, we conclude that A is the generator of $\mathcal{G}_\kappa$. □

An immediate consequence of Theorem 5.2.30 is the next corollary.

Corollary 5.2.31. *Let $(S_\nu(t))_{t\in[0,\tau]}$ a non-degenerate local ν-times integrated semigroup generated by A and $\mathcal{G}_\nu : \mathcal{D}_+ \to \mathcal{B}(X)$ the map defined in Corollary 5.2.25. Then $\mathcal{G}_\nu$ is a $\mathfrak{r}_\nu$-DS generated by A with $\nu > 0$.*

When the generator A defined in Corollary 5.2.31 has dense domain, it is equivalent that A generates an n-times integrated semigroup for some $n \in \mathbb{N}$ and A is the generator of a distribution semigroup in the sense of Lions, see [AEK, Theorem 7.2, Corollary 7.3].

To consider the test-function space $\mathcal{D}_{\kappa^*\infty}$ may be given for a wide class of vector-valued distributions such that different distribution semigroups fall into the scope of this approach.

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